Midterm 2

Math 147, Fall 2018

Name:

Problem 1: Let (X, τ_X) be a topological space.

(a) Let $A \subset X$ be an *open* subset in X ($A \in \tau_X$). Let τ_A be the subspace topology on A. Let $U \subseteq A$. Show that $U \in \tau_A$ if and only if $U \in \tau_X$.

By definition of the subspace topology, $U \in \tau_A$ if and only if $U = V \cap A$ for $V \in \tau_X$. Since A is open in X and V is open in X, $U = V \cap A$ is open in X.

Conversely, if $U \subseteq A$ and $U \in \tau_X$ then $U = U \cap A$ so $U \in \tau_A$.

(b) If $(X, \tau) = (\mathbb{R}, \tau_{Euc})$ is the real line with the Euclidean topology and $A = [0, 1]$, give an example of a subset $U \subset [0, 1]$ which is open in the subspace topology (A, τ_A), but not open in $(\mathbb{R}, \tau_{Euc})$.

Let $U = (\frac{1}{2}, 1]$. Then U is open in the subspace topology because $U = (\frac{1}{2}, \frac{3}{2}) \cap [0, 1]$, but U is not open in $(\mathbb{R}, \tau_{Euc})$ because $1 \in U$ but for any $\varepsilon > 0$, $B_\varepsilon(1) = (1 - \varepsilon, 1 + \varepsilon)$ is not contained in U, so U cannot be open in $\mathbb{R}$.

Problem 2: Let $X = \mathbb{R}^2$ with the Euclidean topology. Define an equivalence relation $\sim$ on X by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $y_1 - x_1 = y_2 - x_2$. Let $Y = \mathbb{R}$ with the Euclidean topology. Construct a map $f : X/_\sim \to Y$ and show that f is *well-defined, continuous and has an inverse.* You do NOT need to prove that f^{-1} is continuous (f^{-1} probably will be continuous, you just do not need to prove it).

Define $f : \mathbb{R}^2/_\sim \to \mathbb{R}$ be defined by

$$f([(x, y)]) = y - x$$

f **is well-defined:** Suppose $[(x_1, y_1)] = [(x_2, y_2)]$ then $y_1 - x_1 = y_2 - x_2$ so

$$f([(x_1, y_1)]) = y_1 - x_1 = y_2 - x_2 = f([(x_2, y_2)])$$

so f is well-defined on equivalence classes.

f **is continuous:** Let $p : \mathbb{R}^2 \to \mathbb{R}^2/_\sim$ be the quotient map. We use the fact that $f : \mathbb{R}^2/_\sim \to \mathbb{R}$ is continuous if and only if $f \circ p$ is continuous.

$$f \circ p((x, y)) = f([(x, y)]) = y - x.$$

Therefore $f \circ p : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial function between Euclidean spaces so it is continuous by standard real analysis arguments.

f **is invertible:** Define $f^{-1} : \mathbb{R} \to \mathbb{R}^2/_\sim$ by

$$f^{-1}(y) = [(0, y)]$$

Then $f(f^{-1}(y)) = f([(0, y)]) = y - 0^2 = y$ and

$$f^{-1}(f([(x, y)])) = f^{-1}(y - x^2) = [(0, y - x^2)]$$

and we can verify that $[(x, y)] = [(0, y - x^2)]$ because $(x, y) \sim (0, y - x^2)$ since $y - x^2 = (y - x^2) - 0^2$.

Problem 3: Let (X, τ_X) and (Y, τ_Y) be Hausdorff topological spaces. Prove that the product space $(X \times Y, \tau_{X \times Y})$ is Hausdorff.

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ where $(x_1, y_1) \neq (x_2, y_2)$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both).

If $x_1 \neq x_2$ then since (X, τ_X) is Hausdorff, there exist open subsets $U_1, U_2 \in \tau_X$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Then $U_1 \times Y \in \tau_{X \times Y}$ and $U_2 \times Y \in \tau_{X \times Y}$. $(x_1, y_1) \in U_1 \times Y$ because $x_1 \in U_1$ and $y_1 \in Y$. $(x_2, y_2) \in U_2 \times Y$ because $x_2 \in U_2$ and $y_2 \in Y$. Finally, $(U_1 \times Y) \times (U_2 \times Y) = \emptyset$ because if $(x, y) \in (U_1 \times Y) \cap (U_2 \times Y)$ then $x \in U_1$, $y \in Y$, and $x \in U_2$, $y \in Y$. Therefore $x \in U_1 \cap U_2$ but $U_1 \cap U_2 = \emptyset$ so this is impossible.

Similarly, if $y_1 \neq y_2$ then there are open sets $V_1, V_2 \in \tau_Y$ such that $y_1 \in V_1$, $y_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then $X \times V_1$ and $X \times V_2$ provide disjoint open subsets containing (x_1, y_1) and (x_2, y_2) respectively.

Problem 4: Let $X = \{a, b, c, d\}$ with the topology

$$\tau_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$$

Prove that (X, τ_2) is connected.

If $A \subset X$ is open and closed, then $A \in \tau_2$ and $X \setminus A \in \tau_2$. Every subset which is open in the topology τ_2 is either the empty set or it contains a. If $a \in A$ then $a \notin X \setminus A$ so the only possibility is that either A or $X \setminus A$ is the empty set. Therefore $A = \emptyset$ or $A = X$. Therefore (X, τ_2) is connected.