Practice Final Exam

Math 147, Fall 2018

Name:

Problem 1: Consider the topology τ_N on $\mathbb{R}$ given by

$$\tau_N = \{(-x,x) | x > 0\} \cup \{\emptyset, \mathbb{R}\}$$

(a) Show that τ_N does give a topology (satisfies the three axioms defining a topology).

1. $\emptyset, \mathbb{R} \in \tau_N$ by definition.
2. If $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of open sets in τ_N then for each α, $U_\alpha = (-x_\alpha, x_\alpha)$ (if $U_\alpha = \emptyset, \mathbb{R}$ we can set $x_\alpha = 0, \infty$). Then

$$\cup_{\alpha \in \mathcal{I}} U_\alpha = (-(\sup_\alpha x_\alpha), \sup_\alpha x_\alpha)$$

   which is in τ_N by definition if $\sup_\alpha x_\alpha$ is finite and is equal to $\mathbb{R}$ if the supremum is infinite (and is the empty set if $\sup_\alpha x_\alpha = 0$ since all the U_α are empty).
3. If $U_1, \cdots, U_n \in \tau_N$ then if any $U_i = \emptyset$, $U_1 \cap \cdots \cap U_n = \emptyset$. Otherwise each $U_i = (-x_i, x_i)$ for $x_i > 0$ or $x_i = \infty$. Let $x = \min\{x_1, \cdots, x_n\}$ then $U_1 \cap \cdots \cap U_n = (-x, x) \in \tau_N$.

(b) Show that $(\mathbb{R}, \tau_N)$ is not Hausdorff

Consider the points $0, 1 \in \mathbb{R}$. If $(\mathbb{R}, \tau_N)$ were Hausdorff, then there would exist open sets $U, V \in \tau_N$ such that $0 \in U$, $1 \in V$ and $U \cap V = \emptyset$. However if V is an open set of τ_N containing 1, then $V = (-x, x)$ where $x > 1$, so $0 \in V$. Therefore if U is an open set containing 0, $U \cap V$ cannot be empty.

(c) Show that $(\mathbb{R}, \tau_N)$ is connected.

The open sets in $(\mathbb{R}, \tau_N)$ are the subsets in the topology described above in the problem statement. The closed sets in $(\mathbb{R}, \tau_N)$ are the complements of open sets, therefore the closed subsets of $(\mathbb{R}, \tau_N)$ are

$$\{(-\infty, -x] \cup [x, \infty) \mid x > 0\} \cup \{\mathbb{R}, \emptyset\}$$

Since these are all unbounded except for the empty set, and the open sets are all bounded except for $\mathbb{R}$, the only subsets of $(\mathbb{R}, \tau_N)$ which are both open and closed are $\emptyset$ and $\mathbb{R}$.

(d) What is the closure of the set $(3, 4)$ in $(\mathbb{R}, \tau_N)$?

The closure of $(3, 4)$ is the set $(-\infty, -3] \cup [3, \infty)$. This is because the closure of a set A is the intersection of all closed sets containing A. We described all the closed subsets of $(\mathbb{R}, \tau_N)$ above, and the only ones which contain $(3, 4)$ are $\mathbb{R}$ and $(-\infty, -x] \cup [x, \infty)$ when $x \leq 3$. Since $(-\infty, -3] \cup [3, \infty)$ is contained in all of these, and is itself one of these closed subsets, the intersection of all these closed subsets which contain $(3, 4)$ is as claimed.

(e) Show that $(\mathbb{R}, \tau_N)$ is not compact.

Consider the infinite collection of open sets $\{(-n, n)\}_{n\in\mathbb{N}}$ where $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the natural numbers. Then

$$\mathbb{R} \subseteq \cup_{n\in\mathbb{N}}(-n, n)$$

because for every number $x \in \mathbb{R}$, there exists a natural number $N = \lceil |x| \rceil$ such that $N > |x|$ so $x \in (-N, N)$. Therefore $\{(-n, n)\}_{n\in\mathbb{N}}$ is an open cover.

If $(\mathbb{R}, \tau_N)$ were compact then there would be a finite subcover $\{(-n_1, n_1), \cdots, (-n_k, n_k)\}$ such that

$$\mathbb{R} \subseteq (-n_1, n_1) \cup \cdots \cup (-n_k, n_k)$$

but letting $N = \max\{n_1, \cdots, n_k\}$ we find that $(-n_1, n_1)\cup\cdots\cup(-n_k, n_k) = (-N, N)$ so $N+1 \in \mathbb{R}$ is an element which is not covered by this union and we get a contradiction.

(f) Let τ_{Euc} be the Euclidean topology on $\mathbb{R}$ and let $f : (\mathbb{R}, \tau_N) \to (\mathbb{R}, \tau_{Euc})$ be the identity function defined by $f(x) = x$. Show that f is *NOT* continuous.

Suppose f were continuous. Then for every open subset U of $(\mathbb{R}, \tau_{Euc})$, $f^{-1}(U)$ would be open in $(\mathbb{R}, \tau_N)$. Consider the interval $(0, 1)$. This is open in the Euclidean topology on $\mathbb{R}$ because it is the open ball of radius $1/2$ centered at $1/2$. However, $f^{-1}((0, 1)) = (0, 1)$, so if f were continuous, then $(0, 1)$ would be open in τ_N, but it does not have the form of an open set in τ_N because for any open subset $V \in \tau_N$, if $x \in V$ then $-x \in V$, but $1/2 \in (0, 1)$ and $-1/2 \notin (0, 1)$.

Problem 2: Let (X, d) be a metric space.

(a) For each $x \in X$ and $n \in \mathbb{N}$ a positive integer, let $B_{1/n}(x) = \{y \in X \mid d(x, y) < 1/n\}$. Let $\mathcal{B} = \{B_{1/n}(x)\}$ indexed over all $x \in X$ and $n \in \mathbb{N}$. Show that $\mathcal{B}$ is a basis.

We must check two criterion:

1. Let $x \in X$, we need to show that there is a basis element in $\mathcal{B}$ containing it. This is true because we can take for example $n = 2$, and center x, and then $x \in B_{1/2}(x) \in \mathcal{B}$.

2. Next we need to show that if $B_1, B_2 \in \mathcal{B}$ and $z \in B_1 \cap B_2$ then there exists $B_3 \in \mathcal{B}$ such that $z \in B_3 \subseteq B_1 \cap B_2$.

   Let $B_1 = B_{1/n_1}(x_1)$ and $B_2 = B_{1/n_2}(x_2)$. If $z \in B_1 \cap B_2$ then $d(z, x_1) < 1/n_1$ and $d(z, x_2) < 1/n_2$. Let

   $$\varepsilon = \min\{\left(\frac{1}{n_1} - d(z, x_1)\right), \left(\frac{1}{n_2} - d(z, x_2)\right)\}.$$

   Then $\varepsilon > 0$ and $B_\varepsilon(z) \subset B_{1/n_1}(x_1) \cap B_{1/n_2}(x_2)$ because for any $y \in B_\varepsilon(z)$,

   $$d(y, x_1) \leq d(y, z) + d(z, x_1) < \varepsilon + d(z, x_1) \leq \frac{1}{n_1} - d(z, x_1) + d(z, x_1) \leq \frac{1}{n_1}$$

   and

   $$d(y, x_2) \leq d(y, z) + d(z, x_2) < \varepsilon + d(z, x_2) \leq \frac{1}{n_2} - d(z, x_2) + d(z, x_2) \leq \frac{1}{n_2}.$$

   Now choose an integer $N > 1/\varepsilon$ so $1/N < \varepsilon$. Then $B_3 = B_{1/N}(z) \in \mathcal{B}$ and $z \in B_3 \subset B_\varepsilon(z) \subset B_1 \cap B_2$.

(b) Let τ be the topology on X induced by the metric d. Let $x_0 \in X$ be any fixed point. Let $C = \{y \in X \mid d(y, x_0) = 5\}$. Show that C is closed.

We will show that $X \setminus C$ is open in the topology induced by the metric.

$$X \setminus C = \{y \in X \mid d(y, x_0) < 5\} \cup \{y \in X \mid d(y, x_0) > 5\}$$

The subset $U_1 = \{y \in X \mid d(y, x_0) < 5\}$ is open because for any $z \in U_1$, $d(z, x_0) < 5$. Let $\varepsilon = 5 - d(z, x_0)$ then $\varepsilon > 0$ and $B_\varepsilon(z) \subseteq U_1$ because for any $y \in B_\varepsilon(z)$,

$$d(y, x_0) \leq d(y, z) + d(z, x_0) < \varepsilon + d(z, x_0) = 5 - d(z, x_0) + d(z, x_0) = 5$$

The subset $U_2 = \{y \in X \mid d(y, x_0) > 5\}$ is open because for any $z \in U_2$ $d(z, x_0) > 5$, so setting $\varepsilon = d(z, x_0) - 5$ we have $\varepsilon > 0$ and $B_\varepsilon(z) \subset U_2$ because for any $y \in B_\varepsilon(z)$,

$$d(x_0, z) \leq d(x_0, y) + d(y, z).$$

Therefore

$$d(y, x_0) \geq d(x_0, z) - d(y, z) > d(x_0, z) - \varepsilon = d(x_0, z) - d(z, x_0) + 5 = 5$$

Therefore $X \setminus C = U_1 \cup U_2$ is the union of two open sets so it is open therefore C is the complement of an open set so it is closed.

Problem 3: Suppose (X, τ) is a topological space which is *compact*. Prove that if $C \subset X$ is a *closed subset*, then C is compact.

(Let $\{V_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of open subsets of C with the subspace topology such that $C \subset \cup_{\alpha \in \mathcal{I}} V_\alpha$. Then since V_α are open in the subspace topology, for each α there exists $U\alpha$ which is open in X such that $V_\alpha = U_\alpha \cap C$. Therefore...)

We have $\{U_\alpha\}$ a collection of open subsets of X such that $C \subset \cup_{\alpha \in \mathcal{I}} U_\alpha$. Since C is closed, $X \setminus C$ is open. Moreover,

$$X \subseteq (X \setminus C) \cup \cup_{\alpha \in \mathcal{I}} U_\alpha$$

because any point in X is either in C in which case it is contained in some $U\alpha$ or it is not in C in which case it is contained in $X \setminus C$. Therefore, this is an open cover of X so since X is compact, it must have a finite subcover. The finite subcover either has the form:

$$X \subseteq (X \setminus C) \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

or

$$X \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

Either way, every point in C must be in at least one of $U_{\alpha_1}, \cdots, U_{\alpha_n}$ since points in C cannot be in $X \setminus C$. Therefore

$$C \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

so $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ is a finite subcover of the cover $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of C

(and $V_{\alpha_1} = U_{\alpha_1} \cap C, \cdots, V_{\alpha_n} = U_{\alpha_n} \cap C$ is a finite subcover of the cover $\{V_\alpha\}_{\alpha \in \mathcal{I}}$ of C in the subspace topology.)

Problem 4: Let $X = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 4\}$ with the Euclidean topology. Define an equivalence relation on X by $(x,y) \sim (x',y')$ if and only if $x' = cx$ and $y' = cy$ for some $c > 0$. Let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$. Prove that the quotient $X/_\sim$ is homeomorphic to S^1 by defining a map $f : X/\sim \to S^1$ which is well-defined, continuous, and has a continuous inverse.

Let $f : X/_\sim \to S^1$ be defined by

$$f([(x,y)]) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

Then f is well-defined on the equivalence classes because if $(x',y') = (cx,cy)$ for $c > 0$ then

$$f([(cx,cy)]) = \left(\frac{cx}{\sqrt{(cx)^2+(cy)^2}}, \frac{cy}{\sqrt{(cx)^2+(cy)^2}}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) = f([(x,y)])$$

and f is well-defined into the circle S^1 because

$$\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = \frac{x^2+y^2}{x^2+y^2} = 1$$

If $p : X \to X/_\sim$ is the projection map $p((x,y)) = [(x,y)]$, then $f \circ p : X \to S^1$ is given by

$$f \circ p((x,y)) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

This is a continuous function from a domain in $\mathbb{R}^2$ into a subset of $\mathbb{R}^2$ so it is a continuous function from X to S^1 using the subspace topologies. Therefore f is a continuous function using the quotient topology on $X/_\sim$.

f has an inverse function $f^{-1} : S^1 \to X/_\sim$ defined by $f^{-1}((x,y)) = [(x,y)]$. This is in fact an inverse because for $(x,y) \in S^1$,

$$f(f^{-1}((x,y))) = f([(x,y)]) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) = (x,y)$$

because $\sqrt{x^2+y^2} = 1$. Also,

$$f^{-1}(f([(x,y)])) = f^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) = \left[\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)\right] = [(x,y)]$$

taking $c = \frac{1}{\sqrt{x^2+y^2}}$.

To show that f^{-1} is continuous, note that $X \subset \mathbb{R}^2$ is closed and bounded so it is compact. Therefore since $p : X \to X/_\sim$ is a continuous surjective function, $X/_\sim = p(X)$ is compact. Therefore $f : X/_\sim \to S^1$ is a continuous bijective function whose domain is a compact set. Additionally S^1 is Hausdorff because it is a subspace of a metric space. Therefore f^{-1} is continuous so f is a homeomorphism.

Problem 5: Let (X, τ_X) and (Y, τ_Y) be compact topological spaces and let $(X \times Y, \tau_{X\times Y})$ be the product space with the product topology. Let $y_0 \in Y$ be a point. Consider the subspace $X \times \{y_0\} \subset X \times Y$ with the subspace topology τ_{sub}. $(X \times \{y_0\} = \{(x, y_0) \mid x \in X\})$. Prove that (X, τ_X) is homeomorphic to $(X \times \{y_0\}, \tau_{sub})$ by defining a map $f : (X, \tau_X) \to (X \times \{y_0\}, \tau_{sub})$ and showing that it is continuous and has continuous inverse.

Define $f : (X, \tau_X) \to (X \times \{y_0\}, \tau_{sub})$ by $f(x) = (x, y_0)$. Then f is bijective because the inverse is given by $f^{-1}(x, y_0) = x$ (clearly $f(f^{-1}(x, y_0)) = f(x) = (x, y_0)$ and $f^{-1}(f(x)) = f^{-1}(x, y_0) = x$). Therefore, we just need to show that f is continuous and f^{-1} is continuous.

Let $V \subset (X\times\{y_0\}, \tau_{sub})$ be an open subset in the subspace topology. Then $V = U \cap (X\times\{y_0\})$ where U is an open subset of $X\times Y$ with the product topology. Since the basis for the product topology is

$$\mathcal{B} = \{U_1 \times U_2 \mid U_1 \in \tau_X, U_2 \in \tau_Y\}$$

U must be a union of basis elements so

$$U = \cup_{\alpha\in\mathcal{I}} U_1^\alpha \times U_2^\alpha$$

Now, $(U_1 \times U_2) \cap (X \times \{y_0\}) = U_1 \times \{y_0\}$ if $y_0 \in U_2$ and is the empty set if $y_0 \notin U_2$. Therefore

$$V = U \cap (X \times \{y_0\}) = \left(\bigcup_{\alpha\in\mathcal{I}} U_1^\alpha \times U_2^\alpha\right) \cap (X \times \{y_0\}) = \bigcup_{\alpha\in\mathcal{I}} ((U_1^\alpha \times U_2^\alpha) \cap (X \times \{y_0\}))$$

$$= \bigcup_{\alpha\in\mathcal{I} \text{ such that } y_0\in U_2^\alpha} U_1^\alpha \times \{y_0\} = \left(\bigcup_{\alpha\in\mathcal{I} \text{ such that } y_0\in U_2^\alpha} U_1^\alpha\right) \times \{y_0\}$$

Therefore

$$f^{-1}(V) = \left(\bigcup_{\alpha\in\mathcal{I} \text{ such that } y_0\in U_2^\alpha} U_1^\alpha\right)$$

which is a union of open subsets $U_1^\alpha \in \tau_X$ so it is open in X.

To show that $f^{-1} : (X \times \{y_0\}, \tau_{sub}) \to (X, \tau_X)$ is continuous, let $U \subset X$ be an open subset of X. Then $(f^{-1})^{-1}(U) = U \times \{y_0\} = (U \times Y) \cap (X \times \{y_0\})$ which is open in the subspace topology since it is the intersection of an open set in the product topology with the subspace.

Problem 6: Show that $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous function if and only if for every *closed* subset of Y, $C \subset Y$, the preimage $f^{-1}(C)$ is closed in X.

Suppose f is continuous. Then for every open subset $U \subset Y$, $f^{-1}(U)$ is open. Let $C \subset Y$ be a closed subset. Then $Y \setminus C$ is open so $f^{-1}(Y \setminus C)$ is open. Now $x \in f^{-1}(C)$ if and only if $f(x) \in C$, so $x \in X \setminus (f^{-1}(C))$ if and only if $f(x) \in Y \setminus C$. Therefore $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$ so $X \setminus f^{-1}(C)$ is open therefore $f^{-1}(C)$ is closed since its complement is open.

Now suppose we know that for every closed subset $C \subset Y$, $f^{-1}(C)$ is closed and we want to show that f is continuous. Let $U \subset Y$ be an open subset. Then $Y \setminus U$ is a closed subset so $f^{-1}(Y \setminus U)$ is closed by assumption. By the same reasoning as above, $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$ so $X \setminus f^{-1}(U)$ is closed therefore $f^{-1}(U)$ is open.