Practice Midterm 1

Math 147, Fall 2018

Problem 1:

(a) Let (X, τ) be a topological space. Suppose V_α is closed in (X, τ) for each $\alpha \in \mathcal{I}$. Prove that the intersection $\cap_\alpha V_\alpha$ is closed.

V_α is closed if and only if $X \setminus V_\alpha$ is open.

$$X \setminus (\cap_\alpha V_\alpha) = \cup_\alpha (X \setminus V_\alpha)$$

The union of any collection of open sets is open so $X \setminus (\cap_\alpha V_\alpha)$ is an open subset. Therefore $\cap_\alpha V_\alpha$ is closed.

(b) Consider $\mathbb{R}$ with the Euclidean topology. Give an example of a infinite collection of closed subsets V_i such that the infinite union $V = \cup_i V_i$ is not closed. (Prove V is not closed in your example.)

Let $V_i = \{1/i\}$ for $i \in \{1, 2, 3, \cdots\}$. To show that $V = \cup_{i=1}^{\infty} V_i$ is not closed, we show that it does not contain all of its limit points. Observe that $0 \neq 1/i$ for any i. However, we will show that 0 is a limit point of V. To show this we need to verify that for any open subset $U \subset \mathbb{R}$ containing 0, $U \cap V \neq \emptyset$. Let $U \subset \mathbb{R}$ be an open subset containing 0. Then there is an open ball $B_\varepsilon(0) = (-\varepsilon, \varepsilon)$ such that $0 \in (-\varepsilon, \varepsilon) \subseteq U$. Choose an integer $N > 1/\varepsilon$. Then $1/N \in (-\varepsilon, \varepsilon) \cap V$. Therefore $1/N \in U \cap V$ so $U \cap V \neq \emptyset$.

Problem 2: Let S^1 be the unit circle in $\mathbb{R}^2$ with the Euclidean metric topology given by

$$S^1 = \{(x_1, x_2) \mid d_{Euc}((x_1, x_2), (0, 0)) = 1\}.$$

Show that S^1 is a closed subset of $\mathbb{R}^2$.

We show that the complement of S^1 is open. $\mathbb{R}^2 \setminus S^1 = A \cup B$ where

$$A = \{(x_1, x_2) \mid d_{Euc}((x_1, x_2), (0, 0)) < 1\}$$

and

$$B = \{(x_1, x_2) \mid d_{Euc}((x_1, x_2), (0, 0)) > 1\}.$$

We will show that both A and B are open subsets of $\mathbb{R}^2$ with the Euclidean topology, so $A \cup B$ is open so S^1 is closed.

To show that A is open, let $x \in A$. We will use the short-hand letting 0 denote the origin $(0, 0) \in \mathbb{R}^2$. Then $d(x, 0) < 1$. Let $\varepsilon = 1 - d(x, 0) > 0$. We claim that $x \in B_\varepsilon(x) \subset A$. To verify this, let $y \in B_\varepsilon(x)$. Then $d(x, y) < \varepsilon = 1 - d(x, 0)$. By the triangle inequality and symmetry of the metric,

$$d(y, 0) \le d(y, x) + d(x, 0) = d(x, y) + d(x, 0) < \varepsilon + d(x, 0) = 1 - d(x, 0) + d(x, 0) = 1$$

so $d(y, 0) < 1$ so $y \in A$.

To show that B is open, let $x \in B$. Then $d(x, 0) > 1$. Let $\delta = d(x, 0) - 1 > 0$. We claim that $x \in B_\delta(x) \subset B$. To verify this, let $y \in B_\delta(x)$. Then $d(y, x) < \delta = d(x, 0) - 1$. By the triangle inequality, $d(x, 0) \le d(x, y) + d(y, 0)$, so

$$d(y, 0) \ge d(x, 0) - d(x, y) > d(x, 0) - \delta = d(x, 0) - (d(x, 0) - 1) = 1$$

so $d(y, 0) > 1$ so $y \in B$.

Problem 3: We define a topology on $\mathbb{R}$ by $\tau_N = \{(-x, x) \mid x > 0\} \cup \{\emptyset, \mathbb{R}\}$.

(a) Prove that τ_N is a topology (satisfies the 3 axioms).

(1) $\emptyset, \mathbb{R} \in \tau_N$ by definition.

(2) If $U_\alpha \in \tau_N$ for all $\alpha \in \mathcal{I}$ then $U_\alpha = (-x_\alpha, x_\alpha)$ for some $x_\alpha > 0$ or $U_\alpha = \emptyset$ or $\mathbb{R}$. If we take a union of subsets of $\mathbb{R}$ where at least one of the subsets is all of $\mathbb{R}$, the union will be $\mathbb{R}$ which is open. Taking the union with the empty set has no effect, so we can assume that all of the U_α have the form $(-x_\alpha, x_\alpha)$. Then

$$\cup_{\alpha \in \mathcal{I}} U_\alpha = \cup_{\alpha \in \mathcal{I}} (-x_\alpha, x_\alpha) = (-x, x)$$

where $x = \sup_{\alpha \in \mathcal{I}} x_\alpha$, if this supremum is finite. $(-x, x) \in \tau_N$. If $\sup_{\alpha \in \mathcal{I}} x_\alpha = \infty$ then $\cup_{\alpha \in \mathcal{I}} U_\alpha = \mathbb{R}$. In both cases, the union is an element of the topology τ_N.

(3) If $U_1, \cdots, U_n \in \tau_N$ then either $U_i = (-x_i, x_i)$ or $U_i = \emptyset$ or $\mathbb{R}$. If all of the $U_i = (-x_i, x_i)$ or $\mathbb{R}$ then $U_1 \cap \cdots \cap U_n = (-x, x)$ where $x = \min_i \{x_i\}$. If any $U_i = \emptyset$ then $U_1 \cap \cdots U_n = \emptyset$ so in both cases the finite intersection is an open set in τ_N.

(b) What is the closure of the subset $(5, 6)$ in the topology τ_N?

The closure of the subset $(5, 6)$ is the intersection of all closed subsets of $(\mathbb{R}, \tau_N)$ which contain $(5, 6)$. $C \subset \mathbb{R}$ is closed using the topology τ_N if and only if $\mathbb{R} \setminus C \in \tau_N$. Therefore C is closed if and only if $C = (-\infty, x] \cup [x, \infty)$ for some $x > 0$ or $C = \mathbb{R}$ or $\emptyset$. The subsets of this form which contain $(5, 6)$ are $(-\infty, x] \cup [x, \infty)$ for $0 < x \leq 5$ and $\mathbb{R}$. Therefore

$$\overline{(5, 6)} = \cap_{0 < x \leq 5} (-\infty, x] \cup [x, \infty) \cap \mathbb{R} = (-\infty, 5] \cup [5, \infty)$$

Problem 4: Let $\mathcal{B}$ be the collection of open parallelograms in $\mathbb{R}^2$ of the form $B_{a,b,c,d} = \{(x,y) \in \mathbb{R}^2 \mid y+a < x < y+b,\ c < y < d\}$ where $a,b,c,d \in \mathbb{R}$ are constants.

(a) Prove that $\mathcal{B}$ is a basis on $\mathbb{R}^2$ (satisfies the two axioms).

(1) Suppose $(x_0, y_0) \in \mathbb{R}^2$. We will find a basis element which contains it.

Let $c = y_0 - 1$, $d = y_0 + 1$, $a = x_0 - y_0 - 1$ and $b = x_0 - y_0 + 1$. Then $(x_0, y_0) \in B_{a,b,c,d}$ because $y_0 - 1 < y_0 < y_0 + 1$ and $y_0 + (x_0 - y_0 - 1) < x_0 < y_0 + (x_0 - y_0 + 1)$.

(2) Suppose $(x_0, y_0) \in B_1 \cap B_2$ where $B_1 = B_{a_1,b_1,c_1,d_1}$ and $B_2 = B_{a_2,b_2,c_2,d_2}$. We will find (a_3, b_3, c_3, d_3) such that $B_{a_3,b_3,c_3,d_3} = B_{a_1,b_1,c_1,d_1} \cap B_{a_2,b_2,c_2,d_2}$ so we satisfy the second basis axiom: $(x_0, y_0) \in B_3 \subseteq B_1 \cap B_2$. Let $a_3 = \max\{a_1, a_2\}$, $b_3 = \min\{b_1, b_2\}$, $c_3 = \max\{c_1, c_2\}$ and $d_3 = \min\{d_1, d_2\}$. Then $(x, y) \in B_1 \cap B_2$ if and only if $c_1 < y < d_1$ and $c_2 < y < d_2$ and $y + a_1 < x < y + b_1$ and $y + a_2 < x < y + b_2$.

$y > c_1$ and $y > c_2$ if and only if $y > \max\{c_1, c_2\} = c_3$. Similarly, $y < d_1$ and $y < d_2$ if and only if $y < \min\{d_1, d_2\} = d_3$.

In the same way, $x > y + a_1$ and $x > y + a_2$ if and only if $x > y + \max\{a_1, a_2\} = y + a_3$. Finally, $x < y + b_1$ and $x < y + b_2$ if and only if $x < y + \min\{b_1, b_2\} = y + b_3$. This verifies that if $B_3 = B_{a_3,b_3,c_3,d_3}$ then $B_3 = B_1 \cap B_2$ so for any $(x_0, y_0) \in B_1 \cap B_2$, $(x_0, y_0) \in B_3 \subseteq B_1 \cap B_2$.

(b) Prove that the unit square $\{(x,y) \mid 0 < x < 1,\ 0 < y < 1\}$ is open in the topology generated by $\mathcal{B}$.

Let $Q = \{(x,y) \mid 0 < x < 1,\ 0 < y < 1\}$ denote the unit square. We will show that for every point $(x_0, y_0) \in Q$, there is a basis element $B_{a,b,c,d}$ such that $(x_0, y_0) \in B \subset Q$. Let $(x_0, y_0) \in Q$ so $0 < x_0 < 1$ and $0 < y_0 < 1$. Let

$$\varepsilon = \min\{x_0 - 0, 1 - x_0, y_0 - 0, 1 - y_0\}$$

Note that $\varepsilon > 0$ because $(x_0, y_0) \in Q$. Let $a = x_0 - y_0 - \varepsilon/2$, $b = x_0 - y_0 + \varepsilon/2$, $c = y_0 - \varepsilon/2$ and $d = y_0 + \varepsilon/2$. We will check that $(x_0, y_0) \in B_{a,b,c,d} \subset Q$. First, to show $(x_0, y_0) \in B_{a,b,c,d}$ we just check that plugging in (x_0, y_0) satisfies the inequalities $a + y < x < b + y$ and $c < y < d$. Indeed $(x_0 - y_0 - \varepsilon/2) + y_0 < x_0 < (x_0 - y_0 + \varepsilon/2) + y_0$ and $y_0 - \varepsilon/2 < y_0 < y_0 + \varepsilon/2$. so $(x_0, y_0) \in B_{a,b,c,d}$.

Next we show that $B_{a,b,c,d} \subset Q$. Suppose $(x, y) \in B_{a,b,c,d}$. Then $(x_0 - y_0 - \varepsilon/2) + y < x < (x_0 - y_0 + \varepsilon/2) + y$ and $y_0 - \varepsilon/2 < y < y_0 + \varepsilon/2$. Since $\varepsilon/2 < \varepsilon \le y_0$ and $\varepsilon/2 < \varepsilon \le 1 - y_0$, we find $0 < y_0 - \varepsilon/2$ and $y_0 + \varepsilon/2 < 1$. Therefore

$$0 < y_0 - \varepsilon/2 < y < y_0 + \varepsilon/2 < 1$$

so $0 < y < 1$. Next we need to check the x coordinate is between 0 and 1 also. We know that $(x_0 - y_0 - \varepsilon/2) + y < x < (x_0 - y_0 + \varepsilon/2) + y$ and $y_0 - \varepsilon/2 < y < y_0 + \varepsilon/2$. Therefore

$$(x_0 - y_0 - \varepsilon/2) + y_0 - \varepsilon/2 < (x_0 - y_0 - \varepsilon/2) + y < x < (x_0 - y_0 + \varepsilon/2) + y < (x_0 - y_0 + \varepsilon/2) + y_0 + \varepsilon/2$$

Simplifying the outside bounds we get

$$x_0 - \varepsilon < x < x_0 + \varepsilon$$

so since $\varepsilon \leq x_0$ and $\varepsilon \leq 1 - x_0$, we have

$$0 \leq x_0 - \varepsilon < x < x_0 + \varepsilon \leq 1$$

so $0 < x < 1$. Therefore $(x, y) \in Q$ so we conclude that every point in $B_{a,b,c,d}$ is in Q so $B_{a,b,c,d} \subset Q$. Thus we have shown Q is open in the topology generated by this basis.