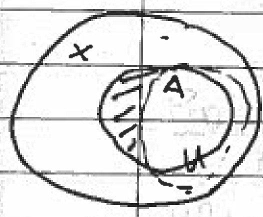
Review

Subspace topology

Defn: If (X,τ) is a topological space and $A \subset X$ a subset the subspace topology τ_A is a topology on A defined by $V \in \tau_A \iff V = U \cap A$ for $U \in \tau_X$.

1 ★ Problem 1: Show that if $A \subset (X,\tau)$ ~~and τ_A is the subspace~~ then $C \subset A$ is closed $\iff$ $C = D \cap A$ for a subset $D \subset X$ closed in τ_X.

Pf: "$\Leftarrow$" If $C = D \cap A$ and D is closed in X
then $X \setminus D$ is open in (X, τ_X)
so $A \cap (X \setminus D)$ is open in (A, τ_A)
$\Rightarrow A \cap (X \setminus D) = A \setminus (D \cap A)$ is open in (A, τ_A)
$\Rightarrow D \cap A = C$ is closed in A.



"$\Rightarrow$" Suppose $C \subset A$ is closed in A
then $A \setminus C$ is open in A
$\Rightarrow A \setminus C = U \cap A$ for some $U \in \tau_X$
$\Rightarrow \underbrace{(X \setminus U)}_{\text{closed in } X \text{ since } U \text{ open in } X} \cap A = C$ ~~because~~ because a point of A is not in $C \iff$ it is in U and A

Problem 2:

~~Suppose (X,τ) is a metric space with~~

Suppose $A \subset (X,\tau)$ is closed. Show that $C \subset A$ is closed in A if and only if C is closed in X.

Pf: By the previous problem, C is closed in A $\iff$ $C = D \cap A$ for D closed in X.

"$\Rightarrow$" Since A is closed in X, and D is closed in X, $A \cap D$ is closed in X.
$\Rightarrow$ If C is closed in A then C is closed in X

"$\Leftarrow$" If C is closed in X and $C \subset A$ then $C = C \cap A$ so C is closed in A.

<u>Problem 3</u>: If (X,τ) is a metric space with the induced metric topology and $A\subset X$ then the subspace topology on A is the same as the metric topology on A.

<u>Pf:</u> Let τ_{sub} be the subspace topology and τ_d be the topology on A induced by the metric d restricted to A.

Then $U\in\tau_d \iff \forall\, u\in U\ \exists\, \varepsilon>0$ s.t.
$$B_\varepsilon(u)\subseteq U$$
where $B_\varepsilon(u)=\{p\in A \mid d(p,u)<\varepsilon\}$.

$U\in\tau_{sub} \iff \exists\, V\subset X$ open s.t. $U=V\cap A$.

$V\subset X$ is open $\iff \forall\, v\in V\ \exists\, \varepsilon>0$ s.t.
$$B_\varepsilon(v)\subset V \text{ where}$$
$$B_\varepsilon(v)=\{p\in X \mid d(p,v)<\varepsilon\}.$$

If $U\in\tau_d$

then for each $u\in U$ let $\varepsilon_u>0$ be such that
$$B^A_{\varepsilon_u}(u)=\{p\in A \mid d(p,u)<\varepsilon_u\}\subseteq U.$$

Let $V=\bigcup_{u\in U} B^X_{\varepsilon_u}(u) = \bigcup_{u\in U}\{p\in X \mid d(p,u)<\varepsilon_u\}$

Then V is open in (X,τ) because it is the union of open sets and $U=V\cap A$ because:

$U\subseteq V\cap A$ since each $u\in U$ is in V and A

and $V\cap A\subseteq U$ because $A\cap B^X_{\varepsilon_u}(u)\subseteq U$ for each $u\in U$.
$$\overset{\shortparallel}{\{p\in A \mid d(p,u)<\varepsilon_u\}}$$

$\Rightarrow U\in\tau_{sub}$.

Conversely if $U \in \tau_{sub}$ then $U = V \cap A$.
For each $u \in U$, $u \in V$ so $\exists\, \varepsilon > 0$ s.t.
$$\{p \in X \mid d(p,u) < \varepsilon\} \subseteq V$$
$$\Rightarrow \{p \in X \mid d(p,u) < \varepsilon\} \cap A \subseteq U$$
$$\Rightarrow \underbrace{\{p \in A \mid d(p,u) < \varepsilon\}}_{B^A_\varepsilon(u)} \subseteq U$$
so $U \in \tau_d$. $\blacksquare$

Product Topology

$X_1 \times \dots \times X_n$

Finite products: Basis $\{U_1 \times \dots \times U_n \mid U_i \in \tau_{X_i}\}$

Infinite products: $\prod_{\alpha \in I} X_\alpha$

Basis $\{\prod_{\alpha \in I} U_\alpha \mid U_\alpha = X_\alpha$ for all but finitely many $U_{\alpha_1}, \dots, U_{\alpha_n}$ where $U_{\alpha_i} \in \tau_{X_{\alpha_i}}\}$

Problem: The product of two connected spaces (A, τ_A) and (B, τ_B) is connected.

Pf: $(A \times B, \tau_{A \times B})$ product space.
Recall that $(A \times B, \tau_{A \times B})$ is not connected $\Leftrightarrow \exists\, f : (A \times B, \tau_{A \times B}) \to \{0,1\}$ continuous, nonconstant.

Suppose for contradiction $A \times B$ is not connected
$\Rightarrow \exists\, f : A \times B \to \{0,1\}$ continuous, surjective
Let $b_0 \in B$ be any point. Let $i_{b_0} : A \to A \times B$ be the function $i_{b_0}(a) = (a, b_0)$. The i_{b_0} is continuous because for any basis element $U_A \times U_B$ where $U_A \in \tau_A$, $U_B \in \tau_B$
$$i_{b_0}^{-1}(U_A \times U_B) = \begin{cases} \emptyset & \text{if } b_0 \notin B \\ U_A & \text{if } b_0 \in B \end{cases}$$
Which is open in (A, τ_A) in both cases.

Since $f: A\times B \to \{0,1\}$ is continuous

$f\circ i_{b_0}: A \xrightarrow{i_{b_0}} A\times B \to \{0,1\}$ is continuous

Since A is connected $f\circ i_{b_0}$ is constant for every $b_0 \in B$.

Similarly $j_{a_0}: B \to A\times B$ defined by $j_{a_0}(b) = (a_0, b)$ is continuous and $f\circ j_{a_0}$ is continuous $\Rightarrow$ constant. for each $a_0 \in A$.

Now since $f: A\times B \to \{0,1\}$ is not constant $\exists$ $(a_1, b_1) \in A\times B$ such that $f((a_1,b_1)) = 0$ and $(a_2, b_2) \in A\times B$ such that $f((a_2,b_2)) = 1$

$\to$ but $f(i_{b_1}(a_1)) = f((a_1,b_1))$

since $f\circ i_{b_1}$ is constant $\to$ $\shortparallel$

$f(i_{b_1}(a_2)) = f((a_2,b_1))$

and $f(j_{a_2}(b_1)) = f((a_2,b_1))$

Since $f\circ j_{a_2}$ is constant $\to$ $\shortparallel$

$f(j_{a_2}(b_2)) = f((a_2,b_2))$

So $1 = f((a_2,b_2)) = f((a_2,b_1)) = f((a_1,b_1)) = 0$

which is a contradiction

Product topology

☆ Claim: $\bar{V}_1 \times \bar{V}_2 = \overline{(V_1 \times V_2)}$

2

Pf: ① $\bar{V}_1 \times \bar{V}_2 \subseteq \overline{V_1 \times V_2}$:

Suppose x is a limit point of V_1
and y is a limit point of V_2

Then $\forall\ U_1 \in \tau_X$ s.t. $x \in U_1$ $\quad U_1 \cap V_1 \neq \emptyset$
and $\forall\ U_2 \in \tau_Y$ s.t. $y \in U_2$ $\quad U_2 \cap V_2 \neq \emptyset$

Therefore if U is an open set in $X \times Y$ and $(x,y) \in U$, $\exists$ $U_1 \in \tau_X$ and $U_2 \in \tau_Y$ such that $U_1 \times U_2 \subseteq U$ so $(U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$
since $\exists$ $\tilde{x} \in U_1 \cap V_1$ and $\tilde{y} \in U_2 \cap V_2$
so $(\tilde{x}, \tilde{y}) \in (U_1 \times U_2) \cap (V_1 \times V_2)$

$\Rightarrow (x,y)$ is a limit point of $V_1 \times V_2$.

② $\overline{V_1 \times V_2} \subseteq \bar{V}_1 \times \bar{V}_2$:

Suppose (x,y) is a limit point of $V_1 \times V_2$.
Then for every open set $U \subset X \times Y$ such that $(x,y) \in U$, $\quad U \cap V_1 \times V_2 \neq \emptyset$.

In particular for every $U_1 \in \tau_X$, $U_2 \in \tau_Y$ such that $x \in U_1$ and $y \in U_2$ $\quad U_1 \times U_2 \cap V_1 \times V_2 \neq \emptyset$

$\Rightarrow \exists\ (\tilde{x}, \tilde{y}) \in U_1 \times U_2 \cap V_1 \times V_2$

$\Rightarrow \tilde{x} \in U_1 \cap V_1$ and $\tilde{y} \in U_2 \cap V_2$ $\Rightarrow U_1 \cap V_1 \neq \emptyset$ and $U_2 \cap V_2 \neq \emptyset$

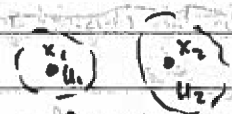
$\Rightarrow x$ is a limit point of V_1 and y is a limit pt of V_2

$\Rightarrow x \in \bar{V}_1$ and $y \in \bar{V}_2$

$\Rightarrow (x,y) \in \bar{V}_1 \times \bar{V}_2$.

Hausdorff, regular, normal

Hausdorff



$U_1 \cap U_2 = \emptyset$

$x_1 \in U_1$, $x_2 \in U_2$

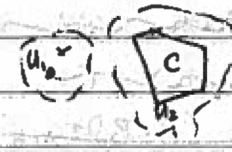
U_1, U_2 open

Examples: Metric spaces

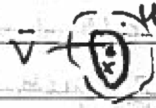
Non-examples: Zariski topology

$\tau = \{(-x, x) \mid x > 0\}$

Regular:



Equivalent to



$\forall x \in X$, U open s.t. $x \in U$

$\exists V$ open s.t. $x \in V$

$\bar{V} \subset U$

Non-example:

Topology on $\mathbb{R}$ where $\{\frac{1}{n}\}$ is closed

$\tau = \{U \mid U \in \tau_{euc}$ or $U = V \setminus \{1/n\}$ for $V \in \tau_{euc}$

Normal

Equivalent to

$\forall C$ c

U o

$C \subset$

$\exists V$ op

s.t. $C \subset V$, $\bar{V} \subset$

Examples:

metric spac

HW example

on $\{1, 2, 3, \ldots$

U open $\Leftrightarrow$

$X \setminus U$ finite

or $1 \in X \setminus U$

3\. **Problem:** If (X, τ_X) and (Y, τ_Y) are regular show $(X \times Y, \tau_{X \times Y})$ is regular.

Pf: Using the equivalent defn of regular:

Let $(x_0, y_0) \in X \times Y$ and $U \subset X \times Y$ be open in $\tau_{X \times Y}$

Then $\exists\ U_1 \in \tau_X$, $U_2 \in \tau_Y$ s.t. $(x_0, y_0) \in \underbrace{U_1 \times U_2}_{\text{basis elt.}} \subseteq U$

$x_0 \in U_1$, $y_0 \in U_2$.

Since (X, τ_X) + (Y, τ_Y) are regular $\exists\ V_1 \in \tau_X$, $V_2 \in \tau_Y$

s.t. $x_0 \in V_1$, $\bar{V_1} \subset U_1$ and $y_0 \in V_2$, $\bar{V_2} \subset U_2$

so $(x_0, y_0) \in V_1 \times V_2$

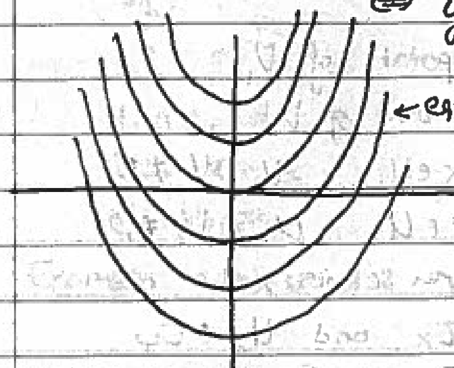
and $\underbrace{\bar{V_1} \times \bar{V_2}}_{\overline{V_1 \times V_2}} \subset U_1 \times U_2$

Quotient Topology

4 $\mathbb{R}^2$ with equivalence relation:

$(x_1, y_1) \sim (x_2, y_2)$

$\Leftrightarrow y_1 - x_1^2 = y_2 - x_2^2$.



Show $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}$.

Define

$f : \mathbb{R}^2/\sim \longrightarrow \mathbb{R}$ by

$f([(x_0, y_0)]) = y - x^2$

<u>f well defined:</u> If $[(x_1, y_1)] = [(x_2, y_2)]$ then

$f([x_1, y_1)]) = y_1 - x_1 = y_2 - x_2 = f([(x_2, y_2)])$ ✓

Let $p : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ be the quotient map then

<u>f continuous:</u> $f \circ p$ is continuous $\Leftrightarrow$ f is continuous.

$f \circ p((x, y_0)) = f([(x, y)]) = y - x^2$

$f \circ p : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial $\Rightarrow$ continuous

$\Rightarrow$ f is continuous

<u>f invertible:</u> Define $f^{-1} : \mathbb{R} \to \mathbb{R}^2/\sim$ by

$f^{-1}(y) = [(0, y)]$

Then $f \circ f^{-1}(y) = f([(0, y)]) = y - 0^2 = y$

and

$f^{-1} \circ f([(x_0, y)]) = f^{-1}(y - x^2) = [(0, y - x^2)]$

and $[(x, y)] = [(0, y \cdot x^2)]$ because

$y - x^2 = (y - x^2) - 0^2$ so $(x, y) \sim (0, y - x^2)$.

Therefore f and f^{-1} are inverse functions

<u>f^{-1} continuous:</u>

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R} \arrow[ur, "g"] \arrow[r, "f^{-1}"] & \mathbb{R}^2/\sim \end{tikzcd}$$

Let $g : \mathbb{R} \to \mathbb{R}^2$ be defined by

$g(y) = (0, y)$

Then $f^{-1} = p \circ g$.

g is continuous and p is continuous $\Rightarrow$ f^{-1} is continuous.

Connectedness

5 (X, τ) is connected iff:

- the only subset $A \subset X$ which are open and closed are $\emptyset$ and X.

(X, τ) is not connected iff:

- there exist nonempty closed subsets $F, G \subset X$ such that $F \cap G = \emptyset$ and $F \cup G = X$

(X, τ) is not connected iff:

there exists a continuous nonconstant function $f : (X, \tau) \to (\{0,1\}, \tau_{discrete})$

Problem: Show $(\mathbb{R}, \tau_{Zariski})$ is connected:

Suppose $A \subset \mathbb{R}$ is open and closed

then either $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite since A is open.

and either $A = \mathbb{R}$ or A is finite since A is closed.

If $A \neq \emptyset$ or $\mathbb{R}$ then A is finite and $\mathbb{R} \setminus A$ is finite

but then $\mathbb{R} = A \cup (\mathbb{R} \setminus A)$ would be the union of two finite sets but $\mathbb{R}$ is infinite $\Rightarrow \Leftarrow$.

Problem: Let $X = \mathbb{N} = \{1, 2, 3, \cdots\}$ τ' The topology defined by

$U \in \tau' \Leftrightarrow$ one or both of the following hold

(a) $X \setminus U$ is finite

(b) $1 \in X \setminus U$

Is (X, τ') connected?

No. $\{2\} \subseteq X$ is open because $1 \in X \setminus U$

$\{2\}$ is closed because $X \setminus \{2\}$ is open because $X \setminus (X \setminus \{2\}) = \{2\}$ is finite.

~~Also~~ $\{2\} \neq \emptyset$ or X so X is not connected.

More generally any finite set that does not contain 1 is open and closed. $\{2\}, \{3\}, \{4\}, \ldots$ are connected components.