HAUSDORFF PROPERTY, LOCALLY EUCLIDEAN, AND QUOTIENT SPACES

MATH 180, WINTER 2023

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include in your notes and presentation, but make sure you at least answer the following questions. You will need to search for some definitions online or in topology textbooks–cite which references you use.

(1) What is the definition of a Hausdorff space? Draw a picture to give intuition for what this definition means.

(2) Prove that $\mathbb{R}^n$ is a Hausdorff space.

(3) Prove that if X is a subset of $\mathbb{R}^n$ with the subspace topology, then X is Hausdorff.

(4) Show that this is an example of a non-Hausdorff space: Consider the real numbers $\mathbb{R}$ with the usual Euclidean topology. Let $\mathbb{Q}$ be the subset of rational numbers. Consider the quotient map $\pi : \mathbb{R} \to Y$ where Y is the quotient of $\mathbb{R}$ under the equivalence relation $\sim$ defined by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. The quotient topology on Y is defined by saying that $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}$ (with the usual Euclidean topology). Show that Y is not a Hausdorff space.

(5) Look up "the line with two origins."
   (a) Explain what the points are in "the line with two origins" and what is the definition of open sets in this space.
   (b) Prove that the line with two origins locally Euclidean (1-dimensional).
   (c) Prove that the line with two origins is NOT Hausdorff.

(6) Find a way to describe "the line with two origins" as a quotient space. Generalize this to define a space (given as a quotient space) which is locally homeomorphic to $\mathbb{R}^n$ (for any given n) and not Hausdorff (prove it satisfies these properties).

(7) Suppose X_1 and X_2 are Hausdorff spaces. Suppose $A_1 \subset X_1$ is a closed subset of X_1 (meaning $X_1 \setminus A_1$ is an open subset of X_1). Similarly suppose $A_2 \subset X_2$ is a closed subset of X_2. Suppose $g : A_1 \to A_2$ is a homeomorphism. Glue A_1 to A_2 using the equivalence relation that says if $x \neq y$ then $x \sim_g y$ if and only if $x \in A_1$ and $y = g(x) \in A_2$ (or vice versa). Prove that the quotient space $(X_1 \sqcup X_2)/\sim_g$ is Hausdorff. Why doesn't this apply for the line with two origins and your generalizations?