# KNOTS AND THE JONES POLYNOMIAL

MATH 180, WINTER 2023

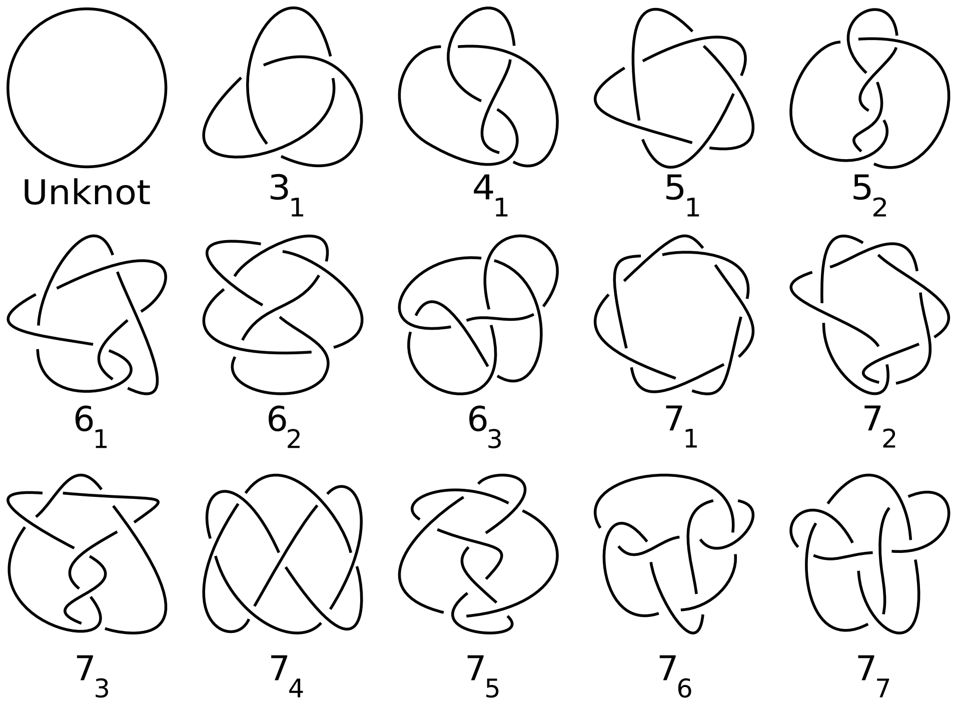
You will need to work together as a group. *You should all work on Problem 1. Each member of the group must be responsible for one full example from problem 2.* Then you can split up problem 3-7 as you wish.

The primary resource for this project is *The Knot Book* by Colin Adams, Chapter 6.1 (page 147-155). *An Introduction to Knot Theory* by Raymond Lickorish Chapter 3, could also be helpful. You may also look at other resources online about knot theory and the Jones polynomial. Make sure to cite the sources you use. If you find it useful and you are comfortable, you can try to write some code to help you with computations.

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include, but make sure you at least go through the following discussion and questions.

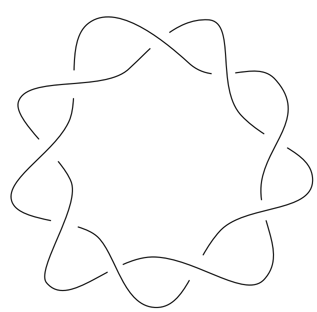
(1) The Bracket Polynomial, Writhe, and the Jones polynomial
   (a) Read through the first four pages of chapter 6.1 of Adams' book where the bracket polynomial is defined. What are the Rules 1,2,3 at the end of this story (only in terms of one variable A)? The discussion in this book works backwards to determine the rules by insisting that the Reidemeister moves hold. Write up an explanation going the other way–namely start with the 3 rules only using the variable A to define the bracket polynomial, and then prove that the the bracket polynomial is unchanged by Reidemeister moves 2 and 3.
   (b) What is the writhe of a knot projection? Explain how to calculate it, and give an informal description of what it is measuring.
   (c) Give an example of a knot diagram and calculate its writhe. Find an example of a different diagram representing an isotopic knot with a different writhe. Find an example of a different knot diagram for a knot which is not isotopic to your first one, but which has the same writhe as your first example.
   (d) How is the Jones polynomial calculated in terms of the bracket polynomial and writhe? Show that if you calculate the Jones polynomial using two diagrams that differ by a Reidemeister 1 move, then the Jones polynomials are the same.
   (e) Calculate the Jones polynomial for the trefoil (the knots $3_1$ in the table below).

(2) Calculate the Jones polynomials for the knots $4_1$, $5_2$, $6_1$, $6_2$, and $6_3$ in the knot table below. Does this show that all of these knots are different?

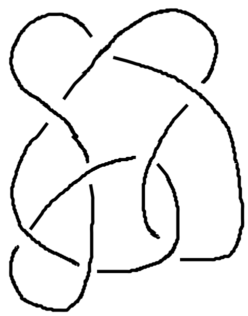


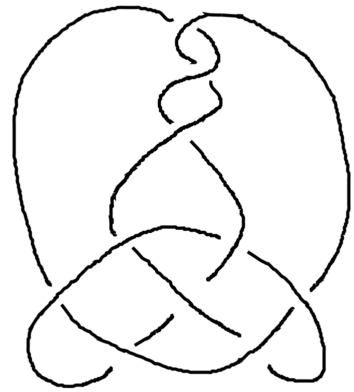

(3) Prove that the knots $7_4$, $7_5$, $7_6$ and $7_7$ in the table above are all different by calculating their Jones polynomials.

(4) A knot which has two strands twisting around in a ring is called a $(2, n)$ torus knot. The knots $3_1$, $5_1$, and $7_1$ in the table above give the $(2, 3)$, $(2, 5)$, and $(2, 7)$ torus knots. Below is the picture of the $(2, 9)$ torus knot. Try to calculate the Jones polynomial for the $(2, n)$ torus knot for a general $n$. Start with the examples with small $n$ pictured here and try to generalize your computations to a strategy that works for a general $n$. (Note that when $n$ is even you get a link with 2 components, whereas when $n$ is odd you get a knot.)

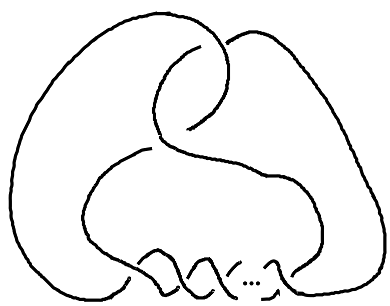


(5) Prove that the Jones polynomials for the following two knots $8_8$ and $10_{129}$ are the same. In fact, these knots are not isotopic, but the Jones polynomial is not enough to tell the difference.





(6) The diagram below shows the general form of a *twist knot* where at the bottom of the diagram there are $n$ crossings. The knots $4_1$, $6_1$ and $7_2$ in the table above give examples of twist knots. Try to calculate the Jones polynomial for a twist knot with $n$ twists at the bottom for a general $n$. Start with calculating cases where $n$ is small and see if you can work out a general strategy which works in general.



(7) Using Chapter 6.2 from Adams *The Knot Book* as a guide, study the properties of the bracket and Jones polynomial for alternating knots. What is the definition of an alternating knot or an alternating knot diagram? What is a reduced or unreduced alternating projection/diagram? Explain the proof of the first Tait conjecture: that two reduced alternating projections of the same knot have the same number of crossings by following the proof in Adams, filling in the gaps by solving the exercises and explaining the proof in your own words. Make sure to give a summary of the key ideas and the overall outline of how the proof works, in addition to explaining details. Do you have any ideas or intuition for how you would try to prove Conjecture 2–that a reduced alternating projection of a knot has the smallest possible number of crossings?