KNOTS AND SEIFERT SURFACES

MATH 180, WINTER 2023

You will need to work together as a group. *Every member of the group must understand Problems 1,2, and 3, and each group member should solve at least one example from problem (4).* You may split up Problems 5-9.

The primary resource for this project is *The Knot Book* by Colin Adams, Chapter 4.3 (page 95-106). *An Introduction to Knot Theory* by Raymond Lickorish Chapter 2, could also be helpful. You may also look at other resources online about knot theory and Seifert surfaces. Make sure to cite the sources you use.

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include, but make sure you at least go through the following discussion and questions.

(1) What is a Seifert surface for a knot? Describe Seifert's algorithm for finding Seifert surfaces: how do you find Seifert circles and how do you use those Seifert circles and twisted bands to build a Seifert surface? How do you know that Seifert's algorithm gives you an orientable (2-sided) surface?

(2) What is the Euler characteristic for a disjoint union of 2-dimensional disks? How does the Euler characteristic change when you glue on a twisted band? What is the Euler characteristic for a Seifert surface built from Seifert's algorithm in terms of the number of Seifert circles and the number of crossings?

(3) We showed that every compact orientable surface without boundary is homeomorphic to either S^2 or the connected sum of g copies of the torus $T^2 \# \cdots \# T^2$. Another name for the connected sum of g copies of the torus, is the *genus g surface*. We consider S^2 to have genus 0. We showed that for two surfaces A and B, the Euler characteristic satisfies

$$\chi(A \# B) = \chi(A) + \chi(B) - 2.$$

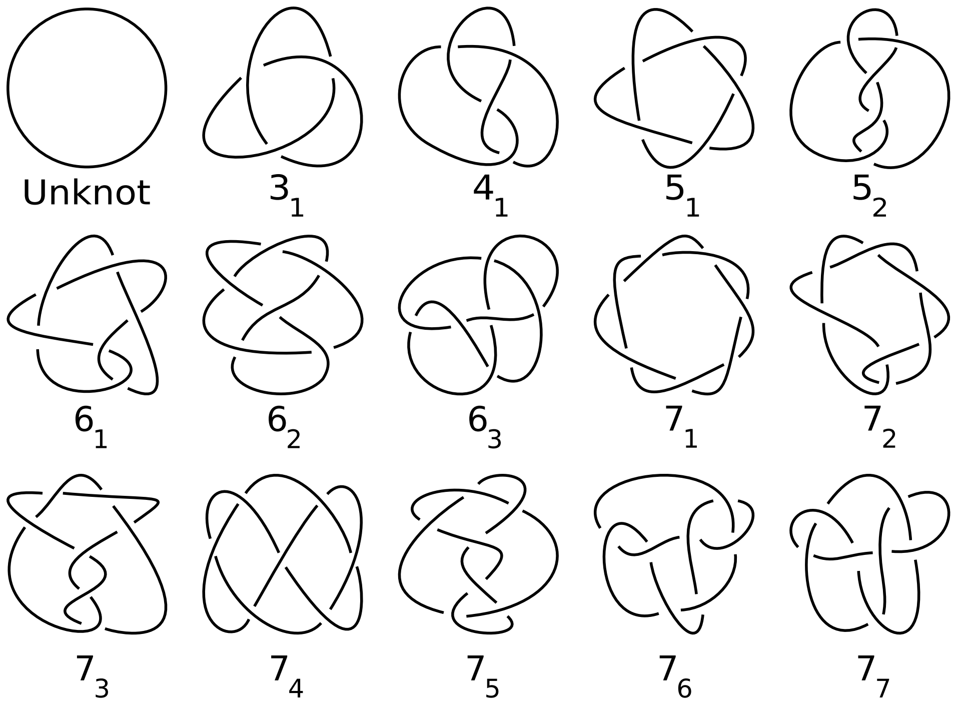
   (a) Using the fact that $\chi(S^2) = 2$ and $\chi(T^2) = 0$, prove by induction that a genus g surface has Euler characteristic $2 - 2g$.

   (b) Consider a surface A, and suppose D is an embedding of a closed disk into A. Let D° be the interior of the disk. Show that

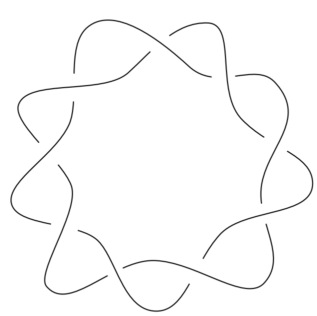
$$\chi(A \setminus D^\circ) = \chi(A) - 1$$

(c) Conclude that for a genus g surface with one hole, and thus one boundary component, the Euler characteristic is $1 - 2g$.

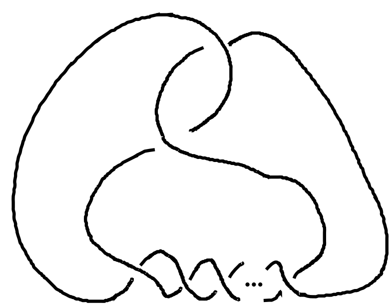
(4) For each of the knots 5_2, 6_1, 6_2, 6_3, and 7_3 shown below, use Seifert's algorithm to determine the Seifert circles and then sketch the Seifert surface. Then calculate the Euler characteristic and use the formula $\chi = 1 - 2g$ to solve for the genus of the Seifert surface.



(5) A knot which has two strands twisting around in a ring is called a $(2, n)$ torus knot (note n must be odd in order for it to be a knot, when n is even you get a link with 2 components). The knots 3_1, 5_1, and 7_1 in the table above give the $(2, 3)$, $(2, 5)$, and $(2, 7)$ torus knots. Below is the picture of the $(2, 9)$ torus knot. Draw the Seifert surfaces for the $(2, 5)$, $(2, 7)$ and $(2, 9)$ torus knots, and calculate the Euler characteristic and genus. Then generalize to describe what a Seifert surface would look like for a $(2, n)$ torus knot for any odd n, and calculate the Euler characteristic and genus.

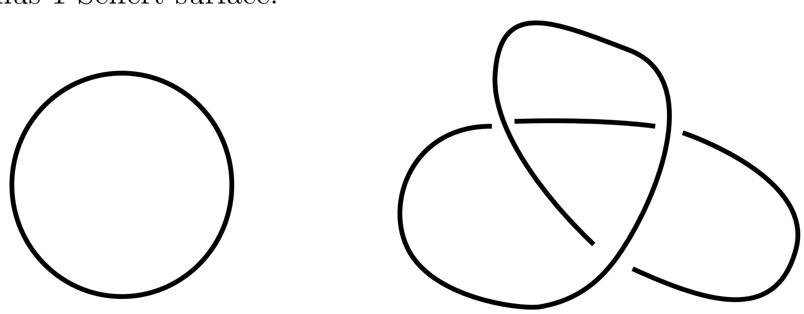


(6) The diagram below shows the general form of a *twist knot* where at the bottom of the diagram there are n crossings. The knots 4_1, 6_1 and 7_2 in the table above give examples of twist knots. What do the Seifert surfaces look like for all twist knots? Calculate their Euler characteristic to prove they are all genus 1.



(7) What is the definition of the genus of a knot? Show that following Seifert's algorithm with different diagrams for isotopic knots may give Seifert surfaces with different genus.

(a) Start by showing that the following two diagrams are isotopic using Reidemeister moves, but show that Seifert's algorithm applied to the diagram on the left gives a genus 0 Seifert surface, whereas when applied to the diagram on the right it gives a genus 1 Seifert surface.



(b) Find another diagram for a knot isotopic to the unknot, such that applying Seifert's algorithm gives a genus 2 Seifert surface.

(c) Can you find a procedure to get as large genus as you choose?

(d) Find an example of another knot, which is not isotopic to the unknot, where you can create two (or more) Seifert surfaces with different genus.

(8) What is the connected sum $J\#K$ of two knots J and K? What is a prime knot? Look closely at the proof of the theorem that

$$g(J\#K) = g(J) + g(K).$$

Explain this proof in your own words and pictures and give examples to show how this proof works. Make sure to fill in any gaps by solving exercises in the proof. Prove as a corollary of the theorem, that genus number 1 knots are prime.

(9) Using use Adams' book Chapter 5.4, what is a braid and what is a closed braid representation of a knot? Draw some examples of closed braids. Give an example of a knot diagram which is not a closed braid and explain why it is not a closed braid. Do exercise 5.16 in Adams' book to find isotopic diagrams to the three knots shown in Figure 5.36 which turn them into closed braids. Now apply Seifert's algorithm to find Seifert surfaces for all of your examples of braids. Is there a pattern? For a general braid, can you tell how many Seifert circles there will be in terms of the number of strands in the braid?