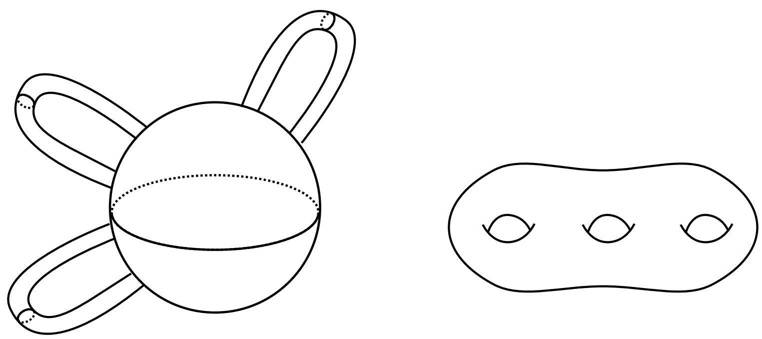
HEEGAARD SPLITTINGS

MATH 180, WINTER 2023

Definition 1. A 3-dimensional genus g handlebody is a solid 3-dimensional manifold with boundary, which can be obtained by attaching g handles to a ball as in the following picture. Equivalently it is the solid 3-dimensional manifold with boundary which is bounded by the standard embedding of the (orientable) genus g surface in $\mathbb{R}^3$.



Notice that a 3-dimensional handlebody can be built from the inside out, by starting with a ball and then attaching the handles, or from the outside in, by starting with the genus g surface, filling in cylinders for the handles, and then filling in by a ball.

Although our intuition in $\mathbb{R}^3$ tell us that there is only one way to fill in a fixed genus g surface with a 3-dimensional handlebody, it is not true that there is only one way to fill in a fixed genus g surface with a 3-dimensional handlebody abstractly. The only case where this is true, is for the genus 0 case. Abstractly, there is only one way to fill in the 2-sphere with a 3-dimensional ball. The formal statement of this is: every homeomorphism of the 2-sphere extends to a homeomorphism of the 3-dimensional ball. Thus, if we tried to glue in the 3-dimensional ball to a fixed 2-sphere by a different homeomorphism of its boundary, we would end up with a homeomorphic object. The reason this is not true for a genus g handlebody is that we have different choices for how to fill in the handles. The way we can specify how we are going to fill in a genus g surface with a handlebody, is to draw the *meridians* of the handles. The meridians form the cores of the cylinders that fill in the handles. The different ways of filling in a genus g surface with a genus g handlebody, give us many options for gluing together two genus g handlebodies along their boundaries to form different 3-dimensional manifolds.

Definition 2. A genus g *Heegaard splitting* of a 3-dimensional manifold Y is a decomposition $Y = H_1 \cup_\partial H_2$ so that H_1 and H_2 are each homeomorphic to genus g handlebodies, and Y

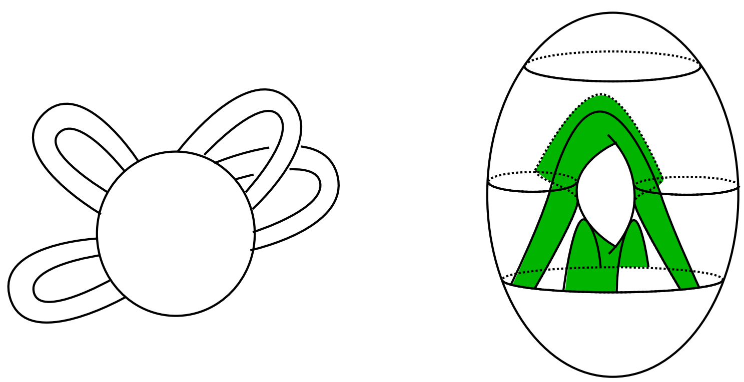
is obtained by gluing together H_1 and H_2 along their boundaries by some homeomorphism from the boundary of H_1 to the boundary of H_2. The genus g surface where H_1 is glued to H_2 is called the *Heegaard surface.*

(1) Describe a genus 0 Heegaard splitting of the 3-dimensional sphere, (obtained by gluing together two 3-dimensional balls along their boundaries).
(2) The next goal is to describe a genus 1 Heegaard splitting of the 3-dimensional sphere. For this, it is useful to think about the 3-dimensional sphere as $\mathbb{R}^3 \cup \{\infty\}$. Embed the torus into $\mathbb{R}^3$ in the standard way. The solid bounded by the torus will be H_1. Explain how to see that the rest of the 3-dimensional sphere (in the complement of H_1) is also a genus 1 3-dimensional handlebody. To do this, think about first putting in a cylinder (corresponding to the handle) in the outside of H_1, and then argue that what is left in the 3-dimensional sphere is homeomorphic to the 3-dimensional ball.
(3) For your genus 1 Heegaard splitting of the 3-dimensional sphere, draw the Heegaard torus. On this torus, draw the meridian for H_1 in red and the meridian for H_2 in blue. How many times does the red meridian intersect the blue meridian on the Heegaard torus?
(4) Now generalize this construction to describe a genus g Heegaard splitting of the 3-dimensional sphere. Draw the genus g Heegaard surface, and draw the g meridians of H_1 in red, and the g meridians of H_2 in blue.
(5) Next, think about the product manifold $S^2 \times S^1$. Let $H_S = S^2 \cap \{z \leq 0\}$ be the southern hemisphere of S^2 and let $H_N = S^2 \cap \{z \geq 0\}$ be the northern hemisphere. Show that $H_S \times S^1 \cup H_N \times S^1$ gives a genus 1 Heegaard splitting of $S^2 \times S^1$.
(6) Next, we will think about the 3-torus, $S^1 \times S^1 \times S^1$. One way to understand this manifold is by taking the cube and gluing together opposite faces (where the gluing map is translating one face to the other with no rotations or flips).
    (a) Draw the cube to represent the 3-torus. Using the identifications of faces indicated above, label which edges get glued to which edges, and which vertices get glued to each other. How many different points are represented by the vertices? How many different line segments are represented by the edges?
    (b) The next goal is to find a Heegaard splitting of the 3-torus. As a first step towards this goal, show that if you have a graph inside a 3-dimensional space, then a 3d thickened neighborhood of the graph is a 3-dimensional handlebody. How can you determine the genus from the graph?
    (c) Show that the union of the vertices and edges of the cube with their identifications, gives a graph inside the 3-torus. If a thickened neighborhood of this graph is a 3-dimensional handlebody as in the previous part, what is the genus of the handlebody?

(d) Given a polyhedron, the vertices and edges form a graph. We form another graph called the *dual graph* using the faces and the 3-dimensional solid as follows. Place one vertex inside the interior of each 3-dimensional solid (there is just one in this case). For each face, place an edge connecting the two vertices on either side of the face (in this case both vertices are the same vertex). Draw the dual graph in the cube.

(e) Show that if you take a 3-dimensional handlebody neighborhood of the graph and another 3-dimensional handlebody neighborhood of the dual graph, then you can expand these two handlebodies until they meet along their boundaries giving a Heegaard splitting of the 3-torus. What is the genus of this Heegaard splitting?

We will take a slight detour to think about 2-dimensional handles and surfaces. A 2-dimensional handlebody is what you obtain by gluing 2-dimensional strips to a 2-dimensional disk as in the following picture. The picture on the right shows how you can build a torus with a hole cut out of the top as a 2-dimensional handlebody.



(7) Using the pictures above as an idea, show that you can build an orientable surface with any genus and any number ≥ 1 of boundary circles. (Also see the Building Surfaces project.)

(8) Using the result of the previous question, show that if F is a genus g surface with at least one boundary circle, then $F \times I$ is a 3-dimensional handlebody.

(9) The final goal is to find a genus g Heegaard splitting for $\Sigma \times S^1$ when Σ is an (orientable) genus g surface. We will do this in steps.

(a) Let $D_1, D_2 \subset \Sigma$ be two disjoint embedded closed 2-dimensional disks in Σ. Let $F_1 = \Sigma \setminus int(D_1)$ and $F_2 = \Sigma \setminus int(D_2)$ be the complements of the interiors of D_1 and D_2 in Σ so F_1 and F_2 are surfaces with boundary. Let $C_U = S^1 \cap \{y \geq 0\}$ be the upper half of the circle, and let $C_L = S^1 \cap \{y \leq 0\}$ be the lower half of the circle. Show that $F_1 \times C_U$ and $F_2 \times C_L$ are each 3-dimensional handlebodies.

(b) Show that $\Sigma \times S^1 = F_1 \times C_U \cup F_2 \times C_L \cup D_1 \times C_U \cup D_2 \times C_L$. We want to combine these four pieces into just two 3-dimensional handlebodies.

(c) Show that $F_1 \times C_U \cup D_2 \times C_L$ is a 3-dimensional handlebody and $F_2 \times C_L \cup D_1 \times C_U$ is a 3-dimensional handlebody. Conclude that $H_1 = F_1 \times C_U \cup D_2 \times C_L$ and $H_2 = F_2 \times C_L \cup D_1 \times C_U$ gives a Heegaard splitting for $\Sigma \times S^1$.

(d) What is the genus g of the Heegaard splitting you constructed?

(e) Think about where the Heegaard surface is in this Heegaard splitting of $\Sigma \times S^1$. Where are the g meridians for H_1? Where are the g meridians for H_2? Can you draw the meridians for H_1 in red and the meridians for H_2 in blue on the same surface?