HOMEWORK 1

MATH 180, WINTER 2023

The goal of this assignment is to review some background we will need to define and work with manifolds.

1. Open subsets of $\mathbb{R}^n$

First, we need to recall the definitions of open balls and open sets in $\mathbb{R}^n$.

Definition 1. The ball of radius ε in $\mathbb{R}^n$ centered at $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, denoted $B_\varepsilon(\mathbf{a})$ is the set of points in $\mathbb{R}^n$ of distance less than ε from $\mathbf{a}$:

$$B_\varepsilon(\mathbf{a}) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid (x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \varepsilon^2\}.$$

In Euclidean space, open sets are defined using ε-balls.

Definition 2. A subset $U \subseteq \mathbb{R}^n$ is *open* if for every point $\mathbf{x} \in U$, there exists a $\varepsilon > 0$ (which usually will depend on $\mathbf{x}$) such that $B_\varepsilon(\mathbf{x}) \subset U$.

(1) Prove that the subset $U \subset \mathbb{R}^3$ defined by

$$U = \{(x_1, x_2, x_3) \mid x_3 < 1\}$$

is open.

(2) Prove that if $U_1, U_2 \subset \mathbb{R}^n$ are two open sets, then their union $U_1 \cup U_2$ is open. In general, prove that if $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ is any collection (potentially infinite) of open sets, then the union $\cup_{\alpha \in \mathcal{I}} U_\alpha$ is open.

(3) Prove that the ball $B_\varepsilon(\mathbf{a})$ is an open subset of $\mathbb{R}^n$.

(4) Prove that any open subset of $\mathbb{R}^n$ is a (possibly infinite) union of balls.

2. The subspace topology

We will usually think of manifolds as subsets of $\mathbb{R}^n$. For example, we can describe the 2-sphere S^2 as the subset of $\mathbb{R}^3$ given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

If we want to focus on S^2 as the space we are interested in instead of $\mathbb{R}^3$, we will need to define what it means to be an open subset of S^2. Specifying the *topology* on S^2 just means specifying what the open subsets of S^2 are. Here we will use the *subspace topology*.

Definition 3. Suppose $A \subseteq \mathbb{R}^n$ is a subset (for example $A = S^2 \subset \mathbb{R}^3$). The *subspace topology* on A is defined by saying a subset $U \subseteq A$ is *open* if and only if there exists a subset $U' \subseteq \mathbb{R}^n$ such that U' is open in $\mathbb{R}^n$ in the usual sense and $U = U' \cap A$.

(5) Apply this definition to prove that the following are open subsets of S^2 as a subspace of $\mathbb{R}^3$:
  (a) The upper hemisphere $H = \{(x, y, z) \in S^2 \mid z > 0\}$
  (b) The complement of the north pole $F = S^2 \setminus N = S^2 \setminus \{(0, 0, 1)\}$
  (c) The complement of the north and south poles $A = S^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$

(6) Consider $S^2 \subset \mathbb{R}^3$ with the subspace topology. Show that $U \subset S^2$ is an open subset using the subspace topology if and only if for every point $\mathbf{x} \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \cap S^2 \subset U$.

3. Continuity

There are many equivalent definitions for continuity of a function, some coming from calculus and others coming from topology. We will start with the usual ε-δ definition of continuity for functions between Euclidean spaces.

Definition 4. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* if for every $\mathbf{a} \in \mathbb{R}^n$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(\mathbf{a})) \subset B_\varepsilon(f(\mathbf{a}))$.

(7) Using the ε-δ definition of continuity, prove that the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\pi(x_1, x_2, x_3) = (x_1, x_2)$ is continuous.

(8) Using the ε-δ definition of continuity, prove that the translation map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + 2, x_2 - 1, x_3 + 5)$ is continuous.