HOMEWORK 2

MATH 180, WINTER 2023

1. Continuity in topology

Recall that the usual ε-δ definition of continuity for functions between Euclidean spaces is as follows.

Definition 1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* if for every $\mathbf{a} \in \mathbb{R}^n$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(\mathbf{a})) \subset B_\varepsilon(f(\mathbf{a}))$.

In topology, we say that a function $f : X \to Y$ is continuous iff whenever $V \subseteq Y$ is an open subset of Y, its preimage $f^{-1}(V) \subset X$ is an open subset. In the next two problems, you'll show that this definition is equivalent to the ε-δ definition.

(1) Suppose you have a function $f : \mathbb{R}^n \to \mathbb{R}^m$ where you know that for any open subset $V \subseteq \mathbb{R}^m$, the preimage $f^{-1}(V)$ is an open set. Show that f is continuous in the ε-δ sense.

(2) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous (in the ε-δ sense), and $V \subseteq \mathbb{R}^m$ is an open set, then the preimage $f^{-1}(V)$ is an open subset of $\mathbb{R}^n$. (Start with what it means that V is open, and see what you can get from knowing that f is ε-δ continuous.)

2. Stereographic projection

Definition 2. The 2-sphere S^2 is the subset of $\mathbb{R}^3$ of points which have distance 1 from the origin:

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Definition 3. *Stereographic projection* from the north pole $N = (0, 0, 1)$, is a function $f : S^2 \setminus N \to \mathbb{R}^2$ defined as follows.

For any point $\mathbf{x} \in S^2 \setminus N$, let ℓ be the unique straight line through $(0, 0, 1)$ and $\mathbf{x}$. Since $\mathbf{x} \neq N$, ℓ intersects the (x_1, x_2)-plane $P = \{x_3 = 0\}$. Then $f(\mathbf{x}) \in \mathbb{R}^2$ is defined to be the point whose coordinates (x_1, x_2) which agree with the (x_1, x_2) coordinates of the intersection point of ℓ with the plane P.

(3) Consider the point $\mathbf{x} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$. Show that $\mathbf{x}$ is a point in S^2. Calculate the linear parametric equation ($x(t) = At + B$, $y(t) = Ct + D$, $z(t) = Et + F$) for the line ℓ that passes through $\mathbf{x}$ and N. Calculate where ℓ intersects the plane P, and write down $f(\mathbf{x})$.

(4) For a general point $\mathbf{x} = (x_0, y_0, z_0) \in S^2 \setminus N$, calculate the linear parametric equation for the line ℓ that passes through $\mathbf{x}$ and N (in terms of the constants x_0, y_0, and z_0). Then calculate where ℓ intersects the plane P, to determine $f(\mathbf{x})$ in terms of x_0, y_0, z_0. This gives a general formula for the function f.

(5) Looking at your function from problem 4, what is the set of points $(x_0, y_0, z_0) \in \mathbb{R}^3$ where this function is defined? Show this function is continuous where it is defined.

3. Inverses and transition functions

You have the definition of $f : S^2 \setminus N \to \mathbb{R}^2$ and your formula from problem 4.

(6) Find a formula for the inverse of f, $g : \mathbb{R}^2 \to S^2 \setminus N$: For $(X, Y) \in \mathbb{R}^2$ find the coordinates of $g(X, Y) \in S^2 \setminus N$. Verify that $f \circ g(X, Y) = (X, Y)$ and $g \circ f(x_0, y_0, z_0) = (x_0, y_0, z_0)$.

(7) Prove that for your inverse function $g : \mathbb{R}^2 \to S^2 \setminus N$, if $U = B_\varepsilon(\mathbf{a}) \cap S^2 \setminus N$ then $g^{-1}(U)$ is an open subset of $\mathbb{R}^2$.

(8) Using the previous problem and the fact that any open subset of $\mathbb{R}^3$ is a union of balls, together with the subspace topology definition of open subsets of $S^2 \setminus N$, show that for any open subset V of $S^2 \setminus N$, $g^{-1}(V)$ is an open subset of $\mathbb{R}^2$. Conclude that g is continuous.

Just as you defined the stereographic projection from the north pole $N = (0, 0, 1)$, we can also define a stereographic projection from the south pole $S = (0, 0, -1)$.

Definition 4. *Stereographic projection* from the *south* pole $S = (0, 0, -1)$, is a function $h : S^2 \setminus S \to \mathbb{R}^2$ defined as follows.

For any point $\mathbf{x} \in S^2 \setminus S$, let ℓ be the unique straight line through $(0, 0, -1)$ and $\mathbf{x}$. Since $\mathbf{x} \neq S$, ℓ intersects the (x_1, x_2)-plane $P = \{x_3 = 0\}$. Then $h(\mathbf{x}) \in \mathbb{R}^2$ is defined to be the point whose coordinates (x_1, x_2) which agree with the (x_1, x_2) coordinates of the intersection point of ℓ with the plane P.

(9) Find a formula as you did in problem 4 for stereographic projection to the south pole: what is $h(x_0, y_0, z_0)$?

(10) Find an inverse function $i : \mathbb{R}^2 \to S^2 \setminus S$ for h.

Now we will put both stereographic projections (from the north and south poles) together.

(11) First determine what is $f((0, 0, -1))$? (Equivalently what is $g^{-1}((0, 0, -1))$?) We will call this point $X = f((0, 0, -1))$.

Then we have a continuous map

$$g : \mathbb{R}^2 \setminus \{X\} \to S^2 \setminus \{S, N\}$$

and another continuous map obtained by restricting h

$$h' : S^2 \setminus \{S, N\} \to \mathbb{R}^2$$

which we can compose.

(12) What is the image of $h' \circ g$?

(13) What is $h' \circ g((2, 0))$? What is $h' \circ g((0, 1/3))$? Where does $h' \circ g$ send the circle of radius r centered at $(0, 0)$?

(14) What is the inverse of $h' \circ g$?

We now have a new way to understand S^2 abstractly without thinking of it inside $\mathbb{R}^3$ as follows: S^2 can be built from two pieces: $S^2 \setminus N$ and $S^2 \setminus S$. Each of these pieces is *homeomorphic* to $\mathbb{R}^2$. Therefore, we can think of S^2 as being built from two copies of $\mathbb{R}^2$. We glue these two pieces of $\mathbb{R}^2$ in the following way:

For every point $\mathbf{a} \in \mathbb{R}^2 \setminus \{X\}$ in the first copy of $\mathbb{R}^2$, we identify this point with $h' \circ g(\mathbf{a})$ in the second copy of $\mathbb{R}^2$.

$$S^2 = \mathbb{R}^2_1 \sqcup \mathbb{R}^2_2 / \sim$$

where the equivalence relation is defined that for any $\mathbf{a} \in \mathbb{R}^2_1 \setminus X_1$ $\mathbf{a} \sim h' \circ g(\mathbf{a}) \in \mathbb{R}^2_2$.