HOMEWORK 3

MATH 180, WINTER 2023

1. Manifolds with boundary

(1) Prove that the closed interval $[0, 1]$ is a 1-dimensional manifold with boundary.
(2) Closed disk
  (a) Show that the hemisphere $H = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$ is a 2-dimensional manifold with boundary.
  (b) Show that the hemisphere H is homeomorphic to the closed disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.
  (c) Use the maps from the previous two parts to prove that the closed disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is a manifold with boundary.
(3) Suppose X is an n-dimensional manifold with boundary. Let ∂X denote the set of points in the boundary of X. Show that ∂X is an $(n-1)$-dimensional manifold.

2. Quotient topology

If we have a space X with an equivalence relation $\sim$, there is a quotient map $q : X \to X/\sim$ where $X/\sim$ is the set of equivalence classes of points in X (if two points are equivalent by the $\sim$ relation we glue them together and consider them the same point). We can turn $X/\sim$ from a set into a space by defining its open sets using the *quotient topology.*

Definition 1. A subset U of $X/\sim$ is open in the quotient topology if and only if $q^{-1}(U) \subset X$ is open in X.

(The notion of open subsets on X *induces* a notion of open subsets on $X/\sim$.)

(4) Let $[0, 1]$ be the closed unit interval. Define an equivalence relation by setting $0 \sim 1$ and all other points are only equivalent to themself. Prove that the quotient space $X = [0, 1]/\sim$ is homeomorphic to the circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$. (Hint: try sending $t \in [0, 1]$ to $(\cos(2\pi t), \sin(2\pi t))$.)
(5) Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the closed 2-dimensional disk. Consider two disjoint copies of D, D_1 and D_2. Each D_i comes with a notion of open subsets from viewing D as a subspace of $\mathbb{R}^2$. We will say a subset U of the disjoint union $D_1 \sqcup D_2$ is open if it is the union of an open subset (possibly empty) of D_1 with an open subset (possibly empty) of D_2. Next, we define an equivalence relation $\sim$ on the space $D_1 \sqcup D_2$ as follows: $(x_a, y_a) \sim (x_b, y_b)$ if and only if one of the following holds
  - $(x_a, y_a) = (x_b, y_b)$ (in particular they are in the same copy of D)
  - $(x_a, y_a) \in D_1$, $(x_b, y_b) \in D_2$, $x_a = x_b$, $y_a = y_b$, and $x_a^2 + y_a^2 = 1$
  - $(x_a, y_a) \in D_2$, $(x_b, y_b) \in D_1$, $x_a = x_b$, $y_a = y_b$, and $x_a^2 + y_a^2 = 1$

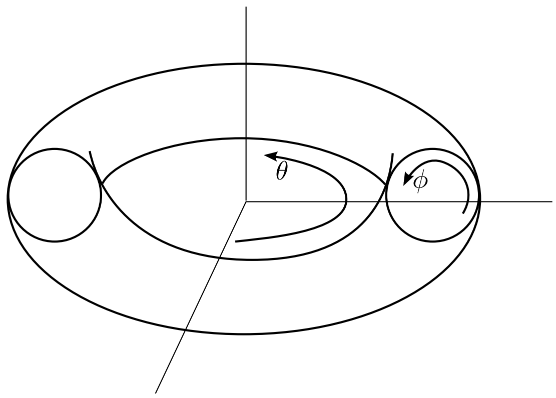
  Prove that the quotient space $X = D_1 \sqcup D_2/\sim$ is homeomorphic to the 2-sphere.

3. The torus

One embedding of the torus T^2 into $\mathbb{R}^3$ is parameterized as follows:

$$T^2 = \{((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi) \in \mathbb{R}^3 \mid \phi, \theta \in \mathbb{R}\}$$

where ϕ and θ parametrize angular coordinates as shown.



(6) Prove that T^2 is a manifold by defining coordinate charts that cover T^2. (Make sure your coordinate domains are open and your coordinate maps are homeomorphisms.)