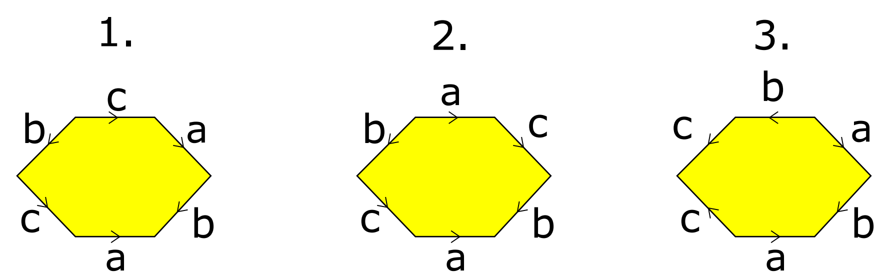
HOMEWORK 4

MATH 180, WINTER 2023

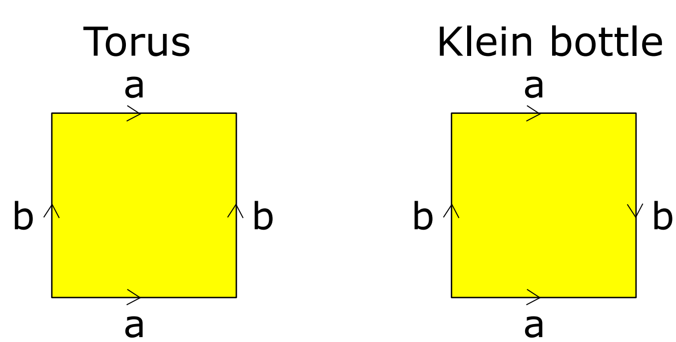
This homework is on polygonal representations of surfaces.

(1) Consider the following three hexagonal representations of surfaces.



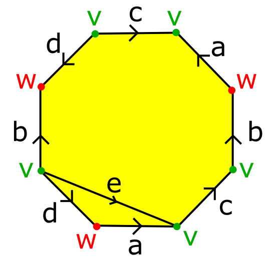
   (a) For each hexagonal representation, how many different points in the glued space are represented by the vertices?
   (b) Notice that hexagonal representation 3. can be folded to become a 4-sided polygon (a square after deforming). What surface does this represent?
   (c) Use cutting and pasting to get hexagonal representations 1. and 2. into positions that allow them to be folded into 4-sided polygons. What surfaces do the results represent?
   (d) Which surfaces are equivalent among those represented by hexagonal representations 1. 2. and 3.?

(2) In class, we showed that some of the ways of labeling the square gave surfaces which could equivalently be represented by bigons. We were left with two types of squares, one giving the torus and the other giving the Klein bottle, which we did not show could be represented by bigons, as shown here:



Prove that in each of these two cases, there is no way to use cutting, pasting, and folding to get an equivalent space represented by a bigon.

(3) Consider the below polygonal representation.



The edge identifications determine which vertices are equivalent to each other. In this case, there are two equivalence classes of vertices, v (green) and w (red). There are 5 representatives of v and 3 representatives of w.

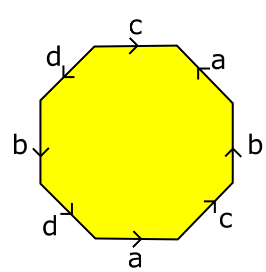
(a) Show that by cutting along e and gluing along a, you get a polygonal representation where there are 6 representatives of v and only 2 representatives of w.

(b) Now take the polygonal representation you created in part (a) and find another cut and paste you can do to change the number of representatives of v to 7 and the number of representatives of w to 1. Then show that you can do a fold to make all the vertices equivalent to each other (v).

(c) Suppose you have any polygonal representation for a manifold (no boundary) where the vertices fall into two equivalence classes v and w. **Prove** that there is always a way to find an equivalent polygonal representation where the number of representatives of w is smaller than in the original presentation, unless the polygon is a bigon.

Therefore, if you start with a polygonal presentation with two equivalence classes of vertices you can always find an equivalent polygonal representation where all the vertices are equivalent to each other. Similarly if you start with a polygonal presentation with more than two equivalence classes of vertices, you can reduce one equivalence class at a time to eventually find a polygonal presentation where all the vertices are equivalent to each other.

(4) We say that two edges with the same label form a *twisted pair* if the arrows both go counterclockwise (or both go clockwise) around the polygon. Consider the below polygonal representation.



Here the edges with labels a, b, and d give twisted pairs. **Find** a sequence of cut-paste moves which transforms this polygonal representation into an equivalent one where any time there is a twisted pair, the two edges with that label are adjacent.