CUT SYSTEMS, CURVE COMPLEXES, AND PANTS GRAPHS OF A SURFACE

MATH 180, WINTER 2023

An isotopy is a 1-parameter continuous deformation. Things which are related by an isotopy are called isotopic. For example, an isotopy between two curves C_0 and C_1 is a 1-parameter family of curves C_t for $t \in [0,1]$ which vary continuously with t.

A simple closed curve in a surface is a closed loop in the surface which does not intersect itself. Equivalently it is the image of a continuous injective map from S^1 into the surface. Within this project we will usually think of two simple closed curves as equivalent if they are isotopic.

Fix an orientable surface Σ of genus greater than 0.

Definition 1. A *cut system* for Σ is a collection of simple closed curves $\{C_1, \ldots, C_n\}$ $(n \geq 1)$ such that

- $\Sigma \setminus \{C_1, \ldots, C_n\}$ is connected
- $\Sigma \setminus \{C_1, \ldots, C_n\}$ is homeomorphic to a 2-sphere with holes (equivalently a disk with holes, or equivalently an open subset of $\mathbb{R}^2$)

(1) Find three different cut systems for the torus. How many simple closed curves are in each cut system?

(2) Find five different cut systems for the genus 2 surface. How many simple closed curves are in each cut system?

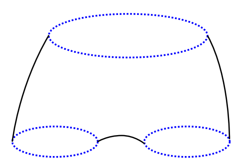
(3) Find seven different cut systems for the genus 3 surface. How many simple closed curves are in each cut system?

An *n-dimensional simplex* is an n-dimensional convex shape with $n+1$ vertices. The 0-dimensional simplex is just a vertex. The 1-dimensional simplex is a line segment connecting two vertices. The 2-dimensional simplex is a triangle connecting 3 vertices. The 3-dimensional simplex is a tetrahedron connecting 4 vertices. Abstractly, an n-dimensional simplex is determined by its vertices (and its dimension). A *face* of an n-dimensional simplex is an $(n-1)$-dimensional simplex which contains all but one of the vertices of the n-dimensional simplex. A *simplicial complex* is a collection of simplices of varying dimensions such that for any simplex in the simplicial complex, all its faces are also in the simplicial complex.

Definition 2. The *curve complex* of Σ is a simplicial complex whose vertices (0-dimensional simplices) are isotopy classes of simple closed curves in Σ. There is a 1-dimensional simplex connecting two vertices, if and only if those vertices correspond to curves which can be isotoped to be disjoint (non-intersecting). More generally, there is an n-dimensional simplex connecting n vertices if and only if those vertices correspond to curves which can be isotoped to be simultaneously all disjoint from each other.

(4) Find five examples of 2-dimensional simplices in the curve complex of the genus 2 surface by specifying the curves corresponding to the three vertices of each 2-dimensional simplex. Can you find any 3-dimensional simplices?
(5) What is the largest dimensional simplex that you can find in the curve complex of the genus 3 surface? Give some examples of simplices of this dimension.

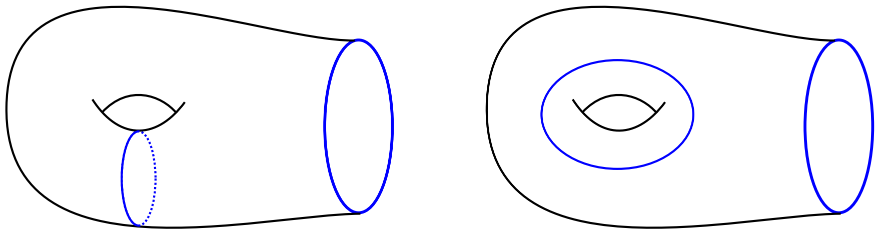
A "pair of pants" is a surface which is homeomorphic to a sphere with three holes as in this figure:

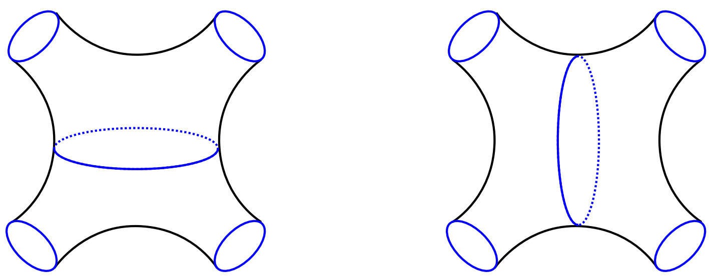


Definition 3. A *pants decomposition* of a surface Σ is a collection of simple closed curves $\{C_1, \ldots, C_n\}$ such that $\Sigma \setminus (C_1 \cup \cdots \cup C_n)$ is homeomorphic to a disjoint union of pairs of pants.

(6) Find 3 different pants decompositions of the genus 2 surface and 5 different pants decompositions of the genus 3 surface.
(7) Show that a collection of curves giving a pants decomposition, always has a subset giving a cut system.
(8) Give a heuristic argument that every simple closed curve in the pair of pants is isotopic to a curve parallel to the boundary of one of the three holes.
(9) Suppose $\{C_1, \ldots, C_n\}$ is a pants decomposition of Σ. Explain how the collection $\{C_1, \ldots, C_n\}$ defines an n-dimensional simplex in the curve complex. Make an argument that says that $\{C_1, \ldots, C_n\}$ is not a face of any $(n+1)$-dimensional simplex using the result of the previous question.
(10) Determine the number of curves n in a pants decomposition of an orientable genus g surface. [Hint: use Euler characteristic.]

The following pictures show a portion of a surface with a pants decomposition. Here each of the boundary components is part of the pants decomposition, so you could insert each piece into a larger surface with a pants decomposition by an embedding (a homeomorphism to its image). Changing the pants decomposition from the top left to the top right is called an *S-move*. Changing the pants decomposition from the bottom left to the bottom right is called an *A-move*. (We always assume that all the curves in parts of the surface which are outside the picture stay exactly the same.)





Definition 4. The *pants graph* of a surface Σ is a graph where the vertices correspond to pants decompositions (up to isotopy), and there is an edge between two vertices if the corresponding pants decompositions are related by an A or an S move.

(11) Fix the surface Σ of genus g and let n be the number of curves in a pants decomposition for Σ determined by your formula above. Show that if two vertices of the pants graph (which by the above correspond to two n-dimensional simplices of the curve complex) are connected by an edge, then that edge corresponds to an $(n-1)$-dimensional simplex in the curve complex which is a face of each of the two n-dimensional simplices of the curve complex. Draw some examples.

(12) Can you show the converse: that every $(n-1)$-dimensional simplex in the curve complex corresponds to an edge in the pants graph? Try some examples and give some ideas.