PROJECTIVE SPACES

MATH 180, WINTER 2023

The *real n-dimensional projective space*, $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} \setminus 0$ by the equivalence relation $(x_1, \cdots, x_{n+1}) \sim (\lambda x_1, \cdots, \lambda x_{n+1})$ for $\lambda \in \mathbb{R} \setminus 0$. A typical way to represent $\mathbb{RP}^n$ is to use *homogeneous coordinates* as follows:

$$\mathbb{RP}^n = \{[x_1 : x_2 : \cdots : x_n, x_{n+1}] \mid (x_1, x_2, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \setminus (0, 0, \cdots, 0, 0)\}$$

where $[x_1 : x_2 : \cdots : x_n : x_{n+1}] = [\lambda x_1 : \lambda x_2 : \cdots : \lambda x_n : \lambda x_{n+1}]$.

(1) Explain how $\mathbb{RP}^n$ can be thought of as the space of lines in $\mathbb{R}^{n+1}$ that pass through the origin.
(2) Consider an equation of the form $a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1} = 0$ for coefficients $a_1, \cdots, a_n, a_{n+1}$. Show that a point $[x_1 : x_2 : \cdots : x_n : x_{n+1}]$ satisfies this equation if and only if $[\lambda x_1 : \lambda x_2 : \cdots : \lambda x_n : \lambda x_{n+1}]$ also satisfies the equation. Conclude that the set of points

$$\{[x_1 : x_2 : \cdots : x_n : x_{n+1}] \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_{n+1}x_{n+1} = 0\}$$

is a well-defined subset of $\mathbb{RP}^n$. Such a subspace is called a *projective hyperplane.*

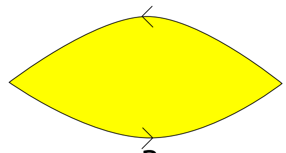
Let $X_1, Y_1 \subset \mathbb{R}P^n$ be the subsets

$$X_1 = \{[x_1 : \cdots : x_{n+1}] \mid x_1 = 1\}$$

$$Y_1 = \{[x_1 : \cdots : x_{n+1}] \mid x_1 = 0\}$$

(3) Show that $\mathbb{R}P^n = X_1 \sqcup Y_1$ (i.e. show that every point in $\mathbb{R}P^n$ is either in X_1 or Y_1 and not both).
(4) Show that X_1 is homeomorphic to $\mathbb{R}^n$.
(5) Show that Y_1 is homeomorphic to $\mathbb{RP}^{n-1}$.
(6) Prove that $\mathbb{RP}^n$ is a manifold of dimension n. (Hint: You gave one coordinate chart using X_1 above. To cover the other points, use an analogous set fixing a different coordinate.)
(7) Think about the specific case where $n = 1$. Prove that $\mathbb{RP}^1$ is homeomorphic to the circle S^1.

(8) Now think about the specific case where $n = 2$. Use the decomposition of $\mathbb{RP}^2$ into $X_1 \cong \mathbb{R}^2$ and $Y_1 \cong \mathbb{RP}^1$ as above (and the analogous decomposition for $\mathbb{RP}^1$), together with the fact that $\mathbb{R}^n$ is homeomorphic to the open n-dimensional ball, to find a way to identify this descrption of $\mathbb{RP}^2$ with the polygonal representation.



Analogously, using the same equations, just changing real numbers to complex numbers, we can define the *complex n-dimensional projective space* $\mathbb{CP}^n$ as the quotient of $\mathbb{C}^{n+1} \setminus 0$ by the equivalence relation $(z_1, \cdots, z_{n+1}) \sim (\lambda z_1, \cdots, \lambda z_{n+1})$ for $\lambda \in \mathbb{C} \setminus 0$. (Here we are using multiplication of complex numbers. Each $z_j = x_j + iy_j$ and $\lambda = \mu + i\nu$. $\lambda z_j = \mu x_j - \nu y_j + i(\nu x_j + \mu y_j)$.) $\mathbb{CP}^n$ also has homogeneous coordinates:

$$\mathbb{CP}^n = \{[z_1 : z_2 : \cdots : z_n, z_{n+1}] \mid (z_1, z_2, \cdots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} \setminus (0, 0, \cdots, 0, 0)\}$$

where $[z_1 : z_2 : \cdots : z_n : z_{n+1}] = [\lambda z_1 : \lambda z_2 : \cdots : \lambda z_n : \lambda z_{n+1}]$.

(9) Use the analogous decomposition of $\mathbb{CP}^n$ into $X_1 \cup Y_1$. Show that X_1 is homeomorphic to $\mathbb{R}^{2n}$ and Y_1 is homeomorphic to $\mathbb{CP}^{n-1}$.