SIMPLICIAL HOMOLOGY

MATH 180, WINTER 2023

The primary resource for this project is **Algebraic Topology** by Allen Hatcher which can be found freely available on Allen Hatcher's website at https://pi.math.cornell.edu/~hatcher/AT/AT.pdf, Chapter 2.1 page 102-107. For some intuition about what the homology of a space is, you may want to read the introduction to Chapter 2 of Hatcher pages 97-101. You may also look at other references you find online, or in Lee's book Chapter 13.

As you research, you may find more examples, definitions, and questions, which you definitely should feel free to include, but make sure you at least go through the following discussion and questions.

(1) What is an n-dimensional simplex? What is the boundary of an n-dimensional simplex? Draw a 0-simplex, 1-simplex, 2-simplex, and 3-simplex and explain what their boundaries are. What is a Δ-complex?

(2) Construct Δ complexes which build the manifolds (or manifolds with boundary) in the following examples (we are always working up to homeomorphism equivalence):
   (a) The circle S^1
   (b) The 2-dimensional sphere S^2
   (c) The 2-dimensional torus T^2
   (d) The 3-dimensional sphere S^3
   (e) The real projective plane $\mathbb{R}P^2$
   (f) The Klein bottle K
   (g) The interval $I = [0, 1]$
   (h) The cylinder (with boundary) C
   (i) The Mobius band (with boundary) M
   (j) The closed 2-dimensional disk (with boundary) D^2

(3) What is the module of n-chains $\Delta_n(X)$ and the boundary map ∂_n? Prove that the image of ∂_{n+1} is in the kernel of ∂_n. What is the definition of the n^{th} homology group of this chain complex $(\Delta_n(X), \partial_n)$?

(4) Using the Δ complexes constructed in Question 2, determine the chain complexes $(\Delta_n(X), \partial_n)$ and the homology groups for each of the spaces in the above examples. Among the orientable examples of manifolds without boundary, if the manifold is k-dimensional what is the rank of its k^{th} homology group? How does this differ from the non-orientable manifolds or manifolds with boundary? Are there any examples in the list which have isomorphic homology groups (which ones)?