MIDTERM (215A) TOPOLOGY

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1. Rules:

- You may not discuss anything about the problems on this midterm with any other person, except to ask questions via email to Laura Starkston.
- You may use Hatcher and your class notes as a reference.
- **You may not access the internet during the time period you are working on the exam.**
- You may quote theorems that were proven in class or are proven in Hatcher in your proofs.
- You may assume that continuous maps between Euclidean spaces are continuous, without proof.

Please include your signature on your exam indicating your agreement to the following statement:

I have carefully read and acknowledged the rules and agree to follow them for this exam. I agree that if I do not follow the rules for this exam, this is considered a breach of the honor code and it will be reported.

Signature: ____

2. Problems

(1) Let $x_0 \in \mathbb{R}\mathrm{P}^2$ be the point $x_0 = [1 : 0 : 0]$. Calculate with proof, $\pi_2(\mathbb{R}\mathrm{P}^2, x_0)$ and $\pi_3(\mathbb{R}\mathrm{P}^2, x_0)$.

(2) Prove that if $f : X \to Y$ is a homeomorphism and $p : E \to Y$ is a fiber bundle, then the pull-back bundle f^*E is homeomorphic to E via a homeomorphism $\Psi : f^*E \to E$ which makes the following diagram commute:

$$\begin{array}{ccc} f^*E & \xrightarrow{\Psi} & E \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

(3) Using the fiber bundle map $p : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$

$$p(x_1, x_2, \cdots, x_{2n+1}, x_{2n+2}) = [x_1 + ix_2 : \cdots : x_{2n+1} + ix_{2n+2}]$$

calculate $\pi_k(\mathbb{C}\mathrm{P}^n, x_0)$ for $2 \leqslant k \leqslant 2n$. Give an explicit map $\phi : S^k \to \mathbb{C}\mathrm{P}^n$ representing each generator for $\pi_k(\mathbb{C}\mathrm{P}^n, x_0)$ for k in this range.

(4) Let B be a topological space and $b_0 \in B$ a base point. Recall the (based) path space is

$$P(B, b_0) = \{\gamma : [0,1] \to B \mid \gamma(0) = b_0\}.$$

As shown in class, the natural map $\phi : P(B, b_0) \to B$ given by $\phi(\gamma) = \gamma(1)$ is a Serre fibration. The fiber

$$\phi^{-1}(b_0) = \Omega(B, b_0) = \{\gamma : [0,1] \to B \mid \gamma(0) = \gamma(1) = b_0\}$$

is called the based loop space of B.

Let c_{b_0} denote the constant path at b_0. Prove that $\pi_n(\Omega(B, b_0), c_{b_0}) \cong \pi_{n+1}(B, b_0)$ for all $n \geqslant 0$.

(5) Show that the spaces S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ have isomorphic homotopy groups for all n, but that there does not exist a map $f : S^2 \to S^3 \times \mathbb{C}\mathrm{P}^\infty$ which is a weak homotopy equivalence inducing the isomorphism.