HOMEWORK 2 (MAT 215A) TOPOLOGY

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(1) Using the decomposition from the previous homework, find cell decompositions of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$, specifying the number of k-cells for each k and writing down the gluing maps $f_i : S^{k-1} \to X^{k-1}$.

Definition: A space X is contractible if the identity map $id_X : X \to X$ ($id_X(x) = x$) is homotopic to a constant map $c : X \to X$ ($c(X) = x_0$ for all $x \in X$).

(2) Recall that the cone of a space X, is the cone of its identity map so: $CX = X \times [0,1]/\sim$ where $(x,1) \sim (x',1)$ for all $x, x' \in X$. Prove that CX is contractible.

(3) Prove that a space is contractible if and only if it is homotopy equivalent to a point.

(4) Let X be a contractible space. Let $A \subset X$ and $r : X \to A$ a retract. Show that A is contractible.

(5) Let $A \subset X$ be a deformation retract of X. Suppose A is contractible. Show that X is contractible.

(6) Prove that if X is homotopy equivalent to Y then ΣX is homotopy equivalent to ΣY.

For the following problems, let (X, x_0) be a pointed space. Let $\gamma : [0,1] \to X$ be any loop with $\gamma(0) = \gamma(1) = x_0$.

(7) Let $e : [0,1] \to X$ be the constant loop $e(t) = x_0$. Write down a homotopy in explicit coordinates from $e \star \gamma$ to γ and from $\gamma \star e$ to γ.

(8) Let $\overline{\gamma}(t) = \gamma(1-t)$. Prove that $\gamma \star \overline{\gamma}$ and is homotopic to the constant loop e (writing the homotopy in explicit coordinates). Conclude that $\overline{\gamma} \star \gamma$ is homotopic to the constant loop as well by substituting γ with $\overline{\gamma}$.

(9) Let $\gamma_1, \gamma_2, \gamma_3 : [0,1] \to X$ all be loops based at x_0. Write down a homotopy in explicit coordinates from $(\gamma_1 \star \gamma_2) \star \gamma_3$ to $\gamma_1 \star (\gamma_2 \star \gamma_3)$.

(10) Show that if $X = \mathbb{R}^n$ and $x_0 \in X$ then $\pi_1(X, x_0)$ is the trivial group.