HOMEWORK 4 (MAT 215A) TOPOLOGY

LAURA STARKSTON

(1) Prove that if X and Y are spaces and $x_0 \in X$, $y_0 \in Y$, then $\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$.

(2) If X and Y are homotopy equivalent (i.e. there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$) show that $\pi_n(X, x_0)$ is isomorphic to $\pi_n(Y, y_0)$.

(3) Suppose there is a retract (not necessarily a deformation retract) $r : X \to A$. Show that the inclusion map $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ is injective, $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is the zero map, and $j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is surjective. You may use the long exact sequence to deduce one of two of these after proving the others. Assume $n \geqslant 2$.

(4) Let $E = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, and $B = \mathbb{C}\mathrm{P}^1$ (which is homeomorphic to S^2). Show that the map

$$p : S^3 \to \mathbb{C}\mathrm{P}^1$$

given by $p(z_1, z_2) = [z_1 : z_2]$ is a fibration whose fiber is S^1. You can also show an analogous thing in general for $p : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$.

(5) $SO(n)$ is the group of $n \times n$ orientation preserving matrices which preserve the standard Euclidean metric on $\mathbb{R}^n$. Since these matrices preserve the vectors of length 1, $SO(n)$ acts on $S^{n-1} \subset \mathbb{R}^n$. Show that the matrices in $SO(n)$ fixing a particular point $x \in S^{n-1}$ form a subset isomorphic to $SO(n-1)$. Verify that $SO(n)$ acts transitively on S^{n-1}. Conclude that there is a quotient map $p : SO(n) \to S^{n-1}$ (where we identify S^{n-1} with the space of orbits of the $SO(n-1)$ action on $SO(n)$). Show that this quotient map is a fibration with fibers $SO(n-1)$.