Lecture 1

Monday, January 4, 2021 1:35 PM

Simplicial Homology

Basic building blocks: simplices ← like cells with extra combinatorial structure

Algebraic structure (addition): just algebraic formalism

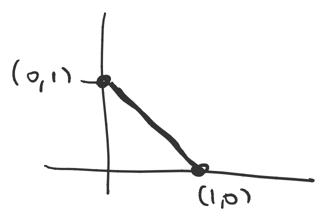
adding ⟷ taking unions

Defn: The n-simplex is

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0 \;\; \forall i \right\}$$

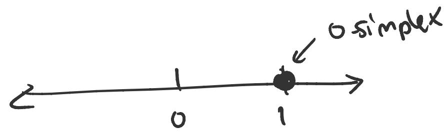
"barycentric coordinates"

Question: What is the 1-simplex?

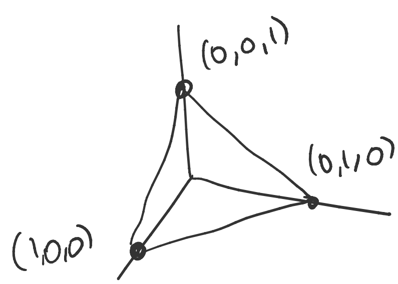


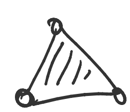
Abstractly

0-simplex



2-simplex





3-simplex

⋮

Extrema / vertices of an n-simplex:

$$\underbrace{(1,0,\dots,0)}_{=v_0},\ \underbrace{(0,1,0,\dots,0)}_{=v_1},\ \dots,\ \underbrace{(0,\dots,0,1)}_{=v_n}$$

$n+1$ vertices (corresponding to the $n+1$ coordinate axes in $\mathbb{R}^{n+1}$)

n-simplex is the <u>convex hull</u> of these vertices $\{v_0, v_1, \dots, v_n\}$

↗ smallest <u>convex set</u> containing these vertices

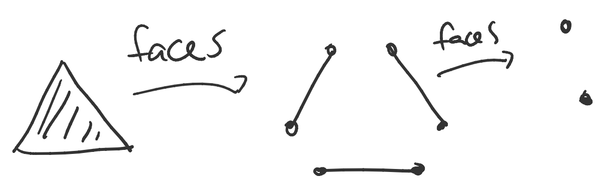
↑ for any 2 pts in the set, the line segment connecting them is contained in the set

Encode the n-simplex by ordered list of vertices $[v_0, v_1, \dots, v_n]$

A <u>face</u> of an n-simplex is the convex hull of all but one of the vertices eg. $[v_0, v_1, \dots, v_{n-1}]$, $[v_0, v_2, \dots, v_n]$

these faces are $(n-1)$-simplices.

(Look at part of n-simplex where one coordinate $=0$)



Want to use simplices to study more interesting spaces

X topological space

<u>Δ-complex</u> ← a structure on a top space X

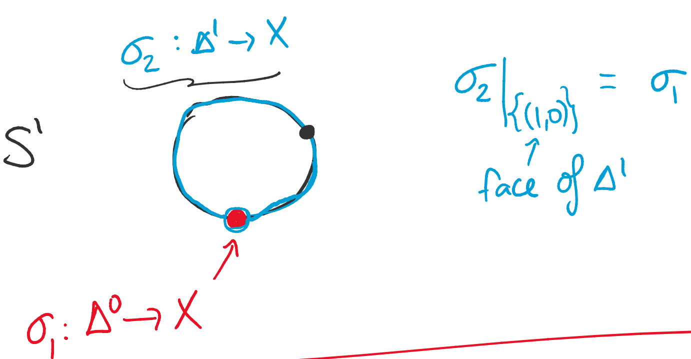
Defn: A Δ-complex structure on space X is a collection of maps $\{\sigma_\alpha : \Delta^n \to X\}_\alpha$ (n can depend on α) such that:

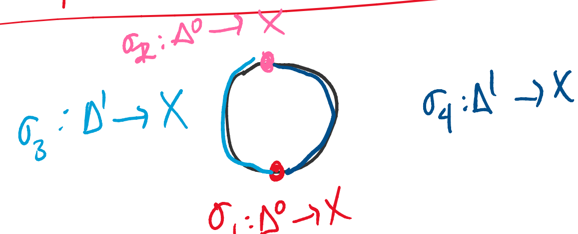
1. $\forall \alpha$ $\sigma_\alpha|_{\mathring{\Delta}^n}$ (restriction to interior) must be injective.
   Every $x \in X$ should be in the image of some $\sigma_\alpha|_{\mathring{\Delta}^n}$
2. $\forall \alpha$ The restriction of $\sigma_\alpha : \Delta^n \to X$ to a face agrees with some other map $\sigma_\beta : \Delta^{n-1} \to X$
3. $A \subset X$ is open $\iff$ $\sigma_\alpha^{-1}(A)$ is open in Δ^n for every α

"Topologies are compatible" The topology on X agrees with the quotient topology obtained by gluing n-simplices together along faces according to the σ_α's

Examples:

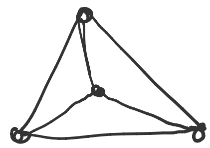
1-dim: $X = S^1$





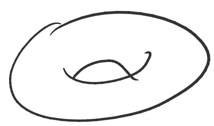
Simplicial complexes -- slightly more restrictive condition

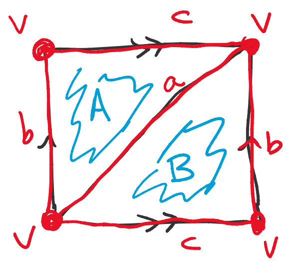
Any n-simplex can be subdivided into many n-simplices



A couple more examples:

T^2 torus





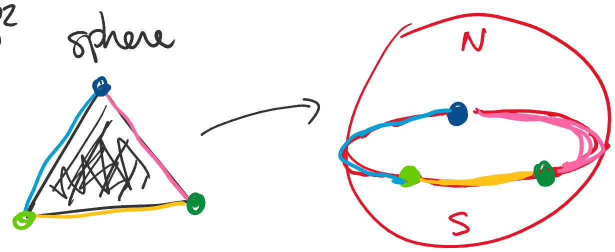
$\sigma_A : \Delta^2 \to T^2$

$\sigma_B : \Delta^2 \to T^2$

$\sigma_a, \sigma_b, \sigma_c : \Delta^1 \to T^2$

$\sigma_v : \Delta^0 \to T^2$

S^2 sphere



2 2-simplices

3 1-simplices

3 0-simplices