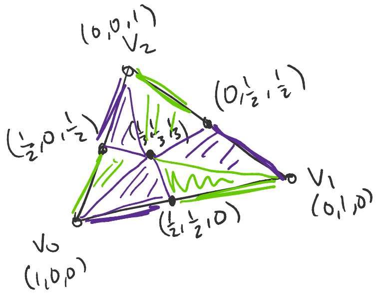
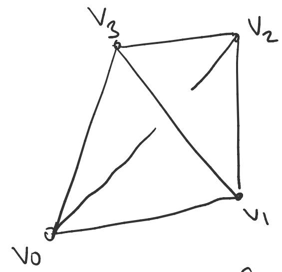
Lecture 10

Friday, January 29, 2021 2:05 PM

HW3 posted





$\Delta^n = \{(t_0, t_1, \dots, t_n) \mid \sum t_i = 1, \ t_i \geq 0\}$

Δ^n: $[v_0, v_1, \dots, v_n]$

Barycentric subdivision of Δ^n:

barycenter of Δ^n: $\left(\frac{1}{n+1}, \frac{1}{n+1}, \dots, \frac{1}{n+1}\right) = \frac{v_0 + v_1 + \dots + v_n}{n+1}$

breaks up Δ^n into simplices: $[b, w_0, \dots, w_{n-1}]$

where b is barycenter of Δ^n and $[w_0, \dots, w_{n-1}]$ are an $(n-1)$-simplex in the barycentric subdiv of a face.

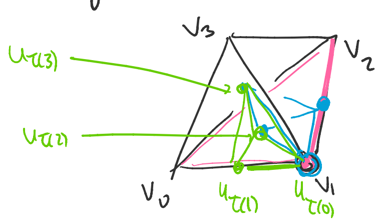
Base case: Δ^0 is its own barycentric subdiv

Alt defn: The n-simplices of barycentric subdiv of Δ^n are in 1-1 corr with sequences $f^0 \subset f_1 \subset \dots \subset f_n$

where f_i is an i-diml face of Δ^n and $f_i \subset f_{i+1}$ $i = 0, \dots, n-1$

τ: $0 \to 1$
$1 \to 0$
$2 \to 2$
$3 \to 3$

$\tilde{\tau}$: $0 \to 1$
$1 \to 2$
$2 \to 0$
$3 \to 3$



$[v_1] \subset [v_1, v_2] \subset [v_1, v_2, v_0] \subset [v_1, v_2, v_0, v_3]$

Take barycenter of each face as vertices of my n-simplex in the subdiv.

The number of n-simplices in $B(\Delta^n)$ is

$$(n+1)\,n\,(n-1)(n-2)\cdots 2\cdot 1 = (n+1)!$$

$\longleftrightarrow$ elements of S_{n+1}

$\tau \in S_{n+1}$

$$B_\tau \leftrightarrow \left([v_{\tau(0)}] \subset [v_{\tau(0)}, v_{\tau(1)}] \subset [v_{\tau(0)}, v_{\tau(1)}, v_{\tau(2)}] \subset \dots \subset [v_{\tau(0)}, \dots, v_{\tau(n)}]\right)$$

Let $u_i^\tau = \dfrac{v_{\tau(0)} + \dots + v_{\tau(i)}}{i+1}$ ← barycenter of $[v_{\tau(0)}, \dots, v_{\tau(i)}]$

$B_\tau = [u_0^\tau, \dots, u_n^\tau]$

Barycentric Transformation: β is a rule which assigns to each top space X

& each integer n, $\beta_n^X : C_n(X) \longrightarrow C_n(X)$

defined by $\beta_n^X\left(\sum K_i \sigma_i\right) := \sum K_i \ \sigma_{i\#}\left(\sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau) B_\tau\right)$

$\sigma_i : \Delta^n \to X$ $\qquad = \sum K_i \sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau) \underbrace{\sigma_i \circ B_\tau}$

restrict σ_i to B_τ ← a piece of barycentric subdiv.

$\sigma_{i\#} : C(\Delta^n) \to C(X)$

$B_\tau : \Delta^n \to \Delta^n$

barycenters

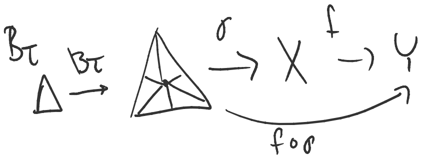
$[\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_n] \longmapsto [u_{\tau(0)}, u_{\tau(1)}, \dots, u_{\tau(n)}]$

Properties: ① $\beta_0^X = id : C_0(X) \to C_0(X)$

② β_n^X are chain maps $\quad \underline{\beta_{n-1}^X \circ d_n} = \underline{d_n \circ \beta_n}$ ← we will check

③ For $f : X \to Y$ $\quad f_\# \circ \beta_n^X = \beta_n^Y \circ f_\#$

$f_\# \circ \beta_n^X(\sigma) = f_\#\left(\sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau) \sigma \circ B_\tau\right) = \sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau) (f \circ \sigma) \circ B_\tau$



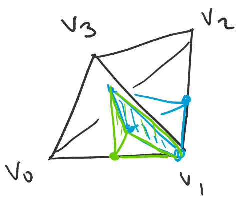
$\beta_n^Y \circ f_\#(\sigma) = \beta_n^Y(f \circ \sigma)$ $\quad //$

To check ②: $d_n \circ \beta_n(\sigma) = d_n\left(\sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau) \sigma \circ B_\tau\right)$

$$= \sum_{i=0}^{n} (-1)^i \sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau) (\sigma \circ B_\tau)\Big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$= \sum_{i=0}^{n} (-1)^i \sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau)\ \sigma\Big|_{[u_{\tau(0)}, u_{\tau(1)}, \dots, \hat{u}_{\tau(i)}, \dots, u_{\tau(n)}]}$$

In these summations, which terms are repeated?



τ 1 2 0 3 face $[u_{\tau(0)}, u_{\tau(2)}, u_{\tau(3)}]$

Shared $\quad ||$

$\tilde{\tau}$ 1 0 2 3 face $[u_{\tilde{\tau}(0)}, u_{\tilde{\tau}(2)}, u_{\tilde{\tau}(3)}]$

τ & $\tilde{\tau}$ differ by transposing 2^{nd} & 3^{rd} elements. $\Rightarrow \operatorname{sgn}(\tau) = -\operatorname{sgn}(\tilde{\tau})$

Face is corresponding to removing $u_{\tau(1)} / u_{\tilde{\tau}(1)}$ $i=1$

$$(-1)^i \operatorname{sgn}(\tau)\, \sigma|_{\text{this face}} + (-1)^i \operatorname{sgn}(\tilde{\tau})\, \sigma|_{\text{this face}} = 0$$

Interior faces of B_τ's are faces of exactly 2 B_τ's.

not on bdry of original Δ^n

Non interior faces of B_τ's only appear once.

Interior faces: how they appear as $(-1)^i \operatorname{sgn}(\tau)\, \sigma|_{[u_{\tau(0)}, \dots, \hat{u}_{\tau(i)}, \dots, u_{\tau(n)}]}$

$(-1)^j \operatorname{sgn}(\tilde{\tau})\, \sigma|_{[u_{\tilde{\tau}(0)}, \dots, \hat{u}_{\tilde{\tau}(j)}, \dots, u_{\tilde{\tau}(n)}]}$

① $i = j$

② $\tau(k) = \tilde{\tau}(k)$ for $k < i$ $\qquad \tau = (i, i+1)\, \tilde{\tau} \qquad \operatorname{sgn}(\tau) = -\operatorname{sgn}(\tilde{\tau})$

$u_{\tau(i+1)} = u_{\tilde{\tau}(i+1)}$

$\shortparallel$

$$\frac{u_{\tau(0)} + \dots + u_{\tau(i-1)} + u_{\tau(i)} + u_{\tau(i+1)}}{i+2} = \frac{u_{\tilde{\tau}(0)} + \dots + u_{\tilde{\tau}(i-1)} + u_{\tilde{\tau}(i)} + u_{\tilde{\tau}(i+1)}}{i+2}$$