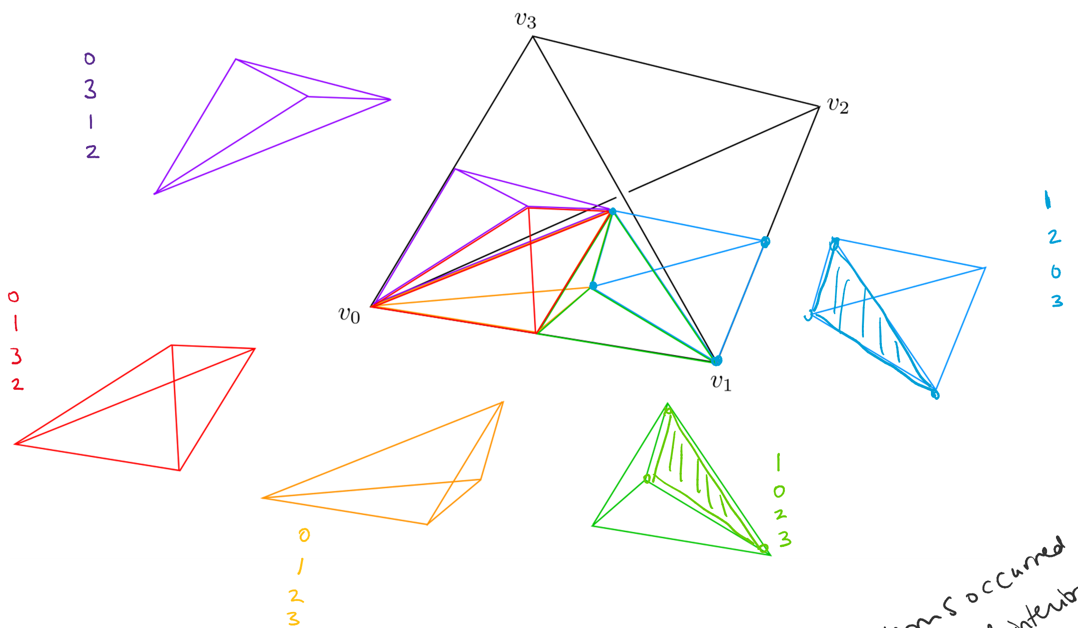
Lecture 11

Monday, February 1, 2021 1:52 PM

HW 3 due Wednesday

$(n+1)!$



$$u_{\tau(j)} = \frac{v_{\tau(0)} + \dots + v_{\tau(j)}}{j+1}$$

$$\beta_n(\sigma) := \sum_{\tau \in S_{n+1}} \operatorname{sgn}(\tau)\, \sigma \circ \beta_\tau$$

Repetition of terms occurred exactly
2 copies of interior faces
2 diff permutations,
i is same in both & $\tau = \tilde{\tau}(i, i+1)$

$(u_{\tau(0)}, \dots, \hat{u}_{\tau(i)}, \dots, u_{\tau(n)})$

$''$

$(u_{\tilde{\tau}(0)}, \dots, \hat{u}_{\tilde{\tau}(j)}, \dots, u_{\tilde{\tau}(n)})$

① $$d_n \circ \beta_n(\sigma) = \underbrace{\sum_{i=0}^{n} \sum_{\tau \in S_{n+1}} (-1)^i \operatorname{sgn}(\tau)\, \sigma|_{[u_{\tau(0)}, \dots, \hat{u}_{\tau(i)}, \dots, u_{\tau(n)}]}}$$

② $$\beta_{n-1} \circ d_n = \beta_{n-1}\left(\sum_{k=0}^{n} (-1)^k \sigma|_{[v_0, \dots, \hat{v}_k, \dots, v_n]} \right)$$

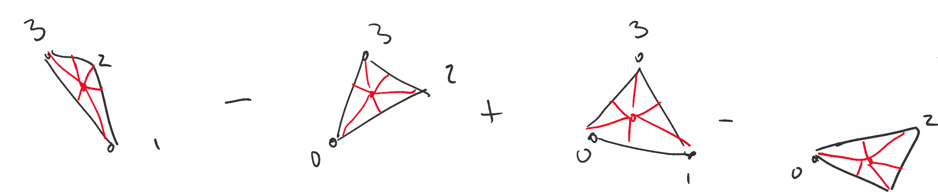
$$= \sum_{k=0}^{n} (-1)^k \left(\sum_{\eta \in S\{0, \dots, \hat{k}, \dots, n\}} \operatorname{sgn}(\eta)\, \sigma|_{[u_{\eta(0)}, \dots, u_{\eta(k-1)}, u_{\eta(k+1)}, \dots, u_{\eta(n)}]} \right)$$

as a permutation of

as a permutation of $\{0, 1, \dots, n-1\}$ add a factor of $(-1)^{n-k}$

Repeated terms cancel out, remaining terms are non-interior faces of barycentric subdivided simplices

$\beta_{n-1} \circ d_n = \beta_{n-1}\Big($



$\Big)$

each term in $\beta_{n-1} d_n(\sigma)$ is a signed exterior face of a β_τ i.e. a nonvanishing term in $d_n \circ \beta_n(\sigma)$

Terms from 2 agree with a term from 1 corr to a $\tau \in S_{n+1} = S\{0,1,\dots,n\}$ which fixes n & permutes according to η

$$\eta \circlearrowright \{0,\dots,k-1,k+1,\dots,n\}$$

$$\eta(i) = \tau(i) \quad 0 \le i \le k-1$$
$$\eta(i+1) = \tau(i) \quad k \le i \le n-1$$
$$\tau(n) = n$$

$$\underline{\operatorname{sgn}(\tau) = (-1)^{n-i} \operatorname{sgn}(\eta)}$$

In 1 $(-1)^n \operatorname{sgn}(\tau)$ is the sign (b/c I've moved the missing vertex from n'th position)
↑ characterizes exterior faces

In 2 $(-1)^i \operatorname{sgn}(\eta)$
$= (-1)^i (-1)^{n-i} \operatorname{sgn}(\tau)$
$= (-1)^n \operatorname{sgn}(\tau)$ so get each term w/ same sign.

<u>Now have established:</u> $\beta_n^X : C_n(X) \to C_n(X)$

- $\beta_{n-1}^X \circ d_n = d \circ \beta_n^X$
- $f: X \to Y$ $\quad f_\# \circ \beta_n^X = \beta_n^Y \circ f_\#$

<u>Lemma</u>: For every X, $\beta_n^X : C_n(X) \to C_n(X)$ are homotopic to the identity via a chain homotopy $h_n^X : C_n(X) \to C_{n+1}(X)$

$$\beta_n^X - \mathrm{id}_{C_n(X)} = h_{n-1}^X \circ d_n + d_{n+1} \circ h_n^X$$

$$\begin{tikzcd} & C_n(X) \arrow[r, "d_n"] \arrow[dl, "h_n"'] \arrow[d, "\beta_n^X - \mathrm{id}"] & C_{n-1}(X) \arrow[dl, "h_{n-1}"] \\ C_{n+1}(X) \arrow[r, "d_{n+1}"'] & C_n(X) & \end{tikzcd}$$

satisfying for any $f: X \to Y$
$$f_\# \circ h_n^X = h_n^Y \circ f_{\#,n}$$

$$\begin{tikzcd} \beta_n^X \circlearrowright C_n(X) \arrow[r, "h_n^X"] \arrow[d, "f_\#"'] & C_{n+1}(X) \arrow[d, "f_\#"] \\ \beta_n^Y \circlearrowright C_n(Y) \arrow[r, "h_n^Y"'] & C_{n+1}(Y) \end{tikzcd}$$

<u>Cor</u>: $(\beta_n^X)_* : H_n(X) \to H_n(X)$ is identity map.

<u>Proof of Lemma</u>: Define $h_n^X : C_n(X) \to C_{n+1}(X)$ inductively in n.

Base: $h_0^X = 0$ for all X.

[illegible] Define $h_n : C_n(X) \to C_{n+1}(X)$ inductively in n.

Base: $h_0^X = 0$ for all X.

Assume h_m^X has been constructed for all $m<n$ s.t.

$$\beta_m^X - id = h_{m-1}^X \circ d_m + d_{m+1} \circ h_m^X$$

$$\& \quad f_\# \circ h_m^X = h_m^Y \circ f_\#$$

Want to find h_n^X: first define $h_n^{\Delta^n}(\sigma_0)$ where $\sigma_0 : \Delta^n \to \underbrace{\Delta^n}_{X}$ is id.

We want

$$d_{n+1} \circ h_n^{\Delta^n}(\sigma_0) = \beta_n^{\Delta^n}(\sigma_0) - \sigma_0 - h_{n-1}^{\Delta^n}(d_n \sigma_0)$$

was defined inductively

$$\underbrace{d_n\left(h_{n-1}^{\Delta^n}(d_n(\sigma_0))\right)} = \underbrace{\beta_{n-1}^{\Delta^n}(d_n(\sigma_0)) - d_n(\sigma_0)} - \underbrace{h_{n-2}^{\Delta}(d_{n-1}(d_n\sigma_0))}_{0}$$

chain htpy cond for $n-1$ applied to $d_n\sigma_0$

$$= \underbrace{d_n\left(\beta_n^{\Delta}(\sigma_0) - \sigma_0\right)}$$

$$0 = d_n\Big(\underbrace{\beta_n^{\Delta}(\sigma_0) - \sigma_0 - h_{n-1}^{\Delta}(d_n(\sigma_0))}\Big)$$

is a cycle in $C_n(\Delta^n)$

For $n>0$
$H_n(\Delta^n) = 0$ because Δ^n is htpy equiv to a point

So any cycle in $C_n(\Delta^n)$ is the boundary of some $(n+1)$ chain in $C_{n+1}(\Delta^n)$ (α)

$$\beta_n^{\Delta}(\sigma_0) - \sigma_0 - h_{n-1}^{\Delta}(d_n(\sigma_0)) = d_{n+1}\alpha$$

Define $\underbrace{h_n^{\Delta}(\sigma_0)} = \alpha$

$\alpha = \sum m_j \alpha_j$

$\alpha_j : \Delta^{n+1} \to \Delta^n$

Extend $h_n^X\left(\sum k_i \sigma_i\right) = \sum k_i \sigma_i(\alpha)$

Extend $h_n^X\left(\sum k_i \sigma_i\right) = \sum k_i \, \sigma_{i\#}(\alpha) = \sum K_i \sum m_j \, \sigma_i \circ \alpha_j$

$\alpha_j : \Delta^{n+1} \to \Delta^n$

$\sigma_i : \Delta^n \to X \qquad \alpha \in C_{n+1}(\Delta^n)$

can check chain htpy cond & naturality under $f: X \to Y$ is clear $\square$