Lecture 12

Wednesday, February 3, 2021 1:54 PM

Barycentric transformation Lemma: For every X, n, $\beta_n^X : C_n(X) \to C_n(X)$ is homotopic to identity.

(Chain htpy $h_n^X : C_n(X) \to C_{n+1}(X)$ $\quad \beta_n^X - id = h_{n-1}^X \circ d_n + d_{n+1} \circ h_n^X$

+ natural: for any $f: X \to Y$ $\quad f_\# h_n^X = h_n^Y \circ f_\#$)

End of proof from last time

Found $h_n^{\Delta^n}(\sigma_0) = \alpha$ with $\beta_n^{\Delta^n}(\sigma_0) - \sigma_0 = h_{n-1}^{\Delta^n} d_n \sigma_0 + d_{n+1} h_n^{\Delta^n}(\sigma_0)$ $\qquad \sigma_0 = id : \Delta^n \to \Delta^n$

Let $h_n^X(\sigma) = \sigma_\#(h_n^{\Delta^n}\sigma_0)$ Then $d_{n+1}(\sigma_\#(h_n^{\Delta^n}\sigma_0)) = \sigma_\#(d_{n+1}(h_n^{\Delta^n}\sigma_0))$

$$
\begin{aligned}
&= \sigma_\#(\beta_n^{\Delta^n}\sigma_0 - \sigma_0 - h_{n-1}^{\Delta^n} d_n \sigma_0) \\
&= \sigma_\# \circ \beta_n^{\Delta}(\sigma_0) - \sigma - \sigma_\# \circ h_{n-1}^{\Delta^n} d_n \sigma_0 \\
&= \beta_n^X(\sigma) - \sigma - h_{n-1}^X \sigma_\# d_n \sigma_0 \\
&= \beta_n^X(\sigma) - \sigma - h_{n-1}^X d_n \sigma
\end{aligned}
$$

So

$$d_{n+1}(h_n^X(\sigma)) + h_{n-1}^X d_n \sigma = \beta_n^X \sigma - \sigma$$

Refinement Lemma:

X space, $\mathcal{U} = \{U_j\}$ open cover of X.

$C_n^{\mathcal{U}}(X) \leftarrow$ generated by $\sigma : \Delta^n \to U_j \subset X$ for some j.

Same differential d_n

Refinement Lemma: The inclusion

$$C_n^{\mathcal{U}}(X) \longrightarrow C_n(X)$$

induces an isomorphism on homology

$$H_n^{\mathcal{U}}(X) \cong H_n(X)$$

Proof: (Fix. $\mathcal{U}$)

Useful facts:

(A) For every chain $c \in C_n(X)$ there exists a suff large N s.t.

$$(\beta_\cdot^X)^N(c) \in C_n^{\mathcal{U}}(X)$$

$$(\beta_n^x)^N(c) \in C_n^{\mathcal{U}}(X)$$

(B) If $d_n c = 0$, $\quad \underbrace{[c - (\beta_n^x)^N(c)]}_{} = 0.$

$(d_n(\beta_n^x)^N c = 0)$

Because β_n^x is homotopic to id

$$(\beta_n^x)_*[c] = [c]$$

(C) If $c \in C_n^{\mathcal{U}}(X)$ and $d_n c = 0$

then $c - (\beta_n^x)^N c = d(\underbrace{h_n^x}_{} c)$ such that $h_n^x c \in C_{n+1}^{\mathcal{U}}(X)$

<u>Why?</u> $\underbrace{i_j : U_j \to X}_{}$ inclusion

$$c = \sum_k m_{j_k} i_{j_k\#}\sigma \in C_n^{\mathcal{U}}(X)$$

$$h_n^x(c) = \sum_k m_{j_k} h_n^x \circ i_{j_k\#}(\sigma) = \sum m_{j_k} i_{j_k\#}(h_n^x(\sigma)) \in C_n^{\mathcal{U}}(X)$$

<u>Surjectivity</u>: For any $[c] \in H_n(X)$

$[c] = [(\beta_n^x)^N(c)]$ and $(\beta_n^x)^N(c) \in C_n^{\mathcal{U}}(X)$

<u>Injectivity</u>: Suppose $[c] = 0$ and $c \in C_n^{\mathcal{U}}(X)$ $d_n c = 0$

Want to show $[c]^{\mathcal{U}} = 0$.

i.e. suppose $c \in C_n^{\mathcal{U}}(X)$ s.t. $c = d_{n+1} c'$ where $c' \in C_{n+1}(X)$

not nec in $\mathcal{U}$

$$c = d_{n+1} c'$$

by chain htpy

$$d_{n+1}(c' - (\beta_n^x)^N c') \overset{\downarrow}{=} d_{n+1}(\cancel{-d_{n+2} h_{n+2}^x c'} - h_n^x d_{n+1} c')$$

$$= -d_{n+1} \circ h_n^x d_{n+1} c'$$

$$= h_n^x(d_n \circ d_{n+1} c')$$

$$= 0$$

$$c = d_{n+1}\left((\beta_n^x)^N c'\right) = d_{n+1} c'$$

$\boxtimes$

Now we know $H_n^{\mathcal{U}}(X) \cong H_n(X)$ (for any $\mathcal{U}$).

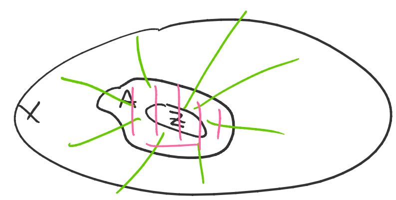
<u>Thm [Excision]</u>: $Z \subset A \subset X$ with $\overline{Z} \subset \mathring{A}$

then $(X - Z,\ A - Z) \xhookrightarrow{\text{inclusion}} (X, A)$ induces an isomorphism on homology

$$H_n(X-Z,\ A-Z) \cong H_n(X, A).$$

<u>Proof</u>: Consider open cover of X $\mathcal{U} = \{U_1, U_2\}$ where $U_1 = X - \overline{Z}$ and $U_2 = \mathring{A}$

$$C_n^{\mathcal{U}}(X, A) = C_n^{\mathcal{U}}(X) / C_n^{\mathcal{U}}(A)$$



$$C_n^{\mathcal{U}}(X-Z, A-Z) = C_n^{\mathcal{U}}(X-Z) / C_n^{\mathcal{U}}(A-Z)$$

<u>Key</u>:
$$C_n^{\mathcal{U}}(X) / C_n^{\mathcal{U}}(A) = C_n^{\mathcal{U}}(X-Z) / C_n^{\mathcal{U}}(A-Z)$$

because any generator of $C_n^{\mathcal{U}}(X)$ either maps <u>entirely into A</u> ← all quotiented out to 0

or <u>maps disjoint from Z</u> ← survive as nonzero generators on both sides

$$C_n(X, A) \xleftarrow{\text{incl}} C_n^{\mathcal{U}}(X, A) = C_n^{\mathcal{U}}(X-Z, A-Z) \xrightarrow{\text{incl}} C_n(X-Z, A-Z)$$

Take homology

$$H_n(X, A) \underset{\star}{\cong} H_n^{\mathcal{U}}(X, A) = H_n^{\mathcal{U}}(X-Z, A-Z) \underset{\star}{\cong} H_n(X-Z, A-Z)$$

$\boxtimes$

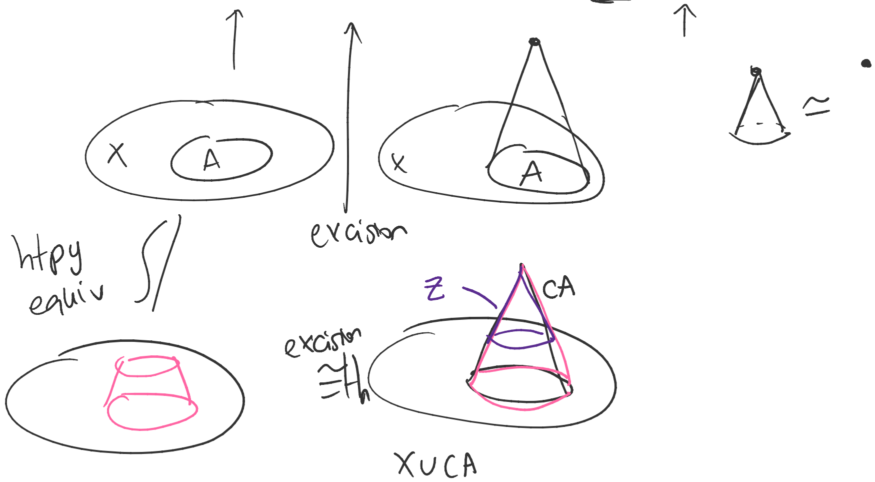
From absolute to relative:

From absolute to relative:

exact
$$\begin{tikzcd}
H_n^U(A) \arrow[r] \arrow[d, "\cong"] & H_n^U(X) \arrow[r] \arrow[d, "\cong"] & H_n^U(X,A) \arrow[r] \arrow[d, "\circledast"] & H_{n-1}^U(A) \arrow[r] \arrow[d, "\cong"] & H_{n-1}^U(X) \arrow[d, "\cong"] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

$\circledast$ is an isomorphism by the five lemma

<u>Cor</u>: $H_n(X,A) \cong H_n(X \cup CA, \underline{CA}) \cong H_n(X \cup CA, pt) \cong \tilde{H}_n(X \cup CA)$

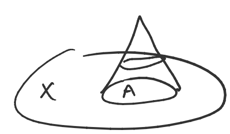


<u>Cor</u>: If $A \subset X$ is closed with a nbhd U in X $\quad A \subset U \subset X$

such that U deformation retracts to A then

$$q: (X \cup CA, CA) \longrightarrow (X \cup CA/CA, CA/CA) \cong (X/A, pt)$$

induces an isomorphism on rel. homology.



$\leadsto H_n(X,A) \cong \tilde{H}_n(X/A)$

$$\begin{tikzcd}
H_n(X \cup CA, CA) \arrow[r, "\cong", "\text{homotopy equiv}"'] \arrow[d, "q_*"] & H_n(X \cup CA, U \cup CA) & H_n(\overbrace{X \cup CA - CA}^{X-A}, \overbrace{U \cup CA - CA}^{U-A}) \arrow[l, "\cong"', "\text{excision}"] \arrow[d, "q_* = id"] \\
H_n(X \cup CA/CA, CA/CA) \arrow[r, "\cong"] & H_n(X \cup CA/CA, U \cup CA/CA) & H_n(X/A - A/A, U/A - A/A) \arrow[l, "\cong"', "\text{excision}"]
\end{tikzcd}$$

$\Rightarrow q_*$ is an isomorphism.