Lecture 13

Friday, February 5, 2021 2:06 PM

<u>New homework 4</u>

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \longrightarrow 0$$

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A)$$

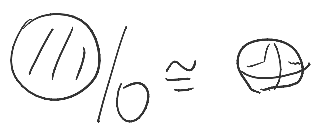
zig zag map in chains

$$\begin{tikzcd} & C_{n-1}(A) \arrow[d, "i_\#"] \\ C_n(X) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X) \\ C_n(X)/C_n(A) & \end{tikzcd}$$

If $A \subset X$ has a nbhd that def retracts to A then

$$H_n(X,A) \cong H_n(X/A, pt) \cong \tilde{H}_n(X/A)$$

In particular, $H_n(D^k, S^{k-1}) \cong H_n(D^k/S^{k-1}, pt) \cong H_n(S^k, pt) \cong \tilde{H}_n(S^k)$



<u>Compute $H_n(D^k, S^{k-1}) \cong \tilde{H}_n(S^k)$ by induction on K:</u>

$$\underset{\substack{\parallel \\ 0}}{H_n(D^k)} \to H_n(D^k, S^{k-1}) \xrightarrow{\cong} H_{n-1}(S^{k-1}) \xrightarrow{0} \underset{\substack{\parallel \\ 0}}{H_{n-1}(D^k)} \quad \text{exactness}$$

isomorphism

D^k is contractible

so if $n>1$ $H_n(D^k) = H_{n-1}(D^k) = 0$

$$H_n(S^k) \cong H_n(D^k, S^{k-1}) \cong H_{n-1}(S^{k-1})$$

<u>Base case</u>: $S^0 = 2$ points $\quad H_0(S^0) \cong \mathbb{Z}^2 \quad H_n(S^0) = 0$ for $n \neq 0$.

$$\underset{\substack{\parallel \\ 0}}{H_1(D^k)} \to H_1(D^k, S^{k-1}) \overset{\partial}{\hookrightarrow} \underset{\substack{\cong \\ \mathbb{Z}^2}}{H_0(S^{k-1})} \xrightarrow{i_*} \underset{\substack{\cong \\ \mathbb{Z}}}{H_0(D^k)} \xrightarrow{0} H_0(D^k, S^{k-1}) \to 0$$

$K = 1$

$(a,b) \longmapsto a+b$

$H_1(D^k, S^{k-1}) \cong im(\partial) = ker(i_*) = \{(a,-a)\} \cong \mathbb{Z}$

$$H_1(D^k, S^{k-1}) \cong im(\partial) = ker(i_*) = \{(a,-a)\} \cong \mathbb{Z}$$

$K=1$ $\quad H_1(D^k, S^{k-1}) \cong \mathbb{Z}$, $\quad H_0(D^k, S^{k-1}) = 0$

$K>1$

$$\begin{array}{ccccccccc} 0 \longrightarrow H_1(D^k, S^{k-1}) & \xrightarrow{\partial} & H_0(S^{k-1}) & \xrightarrow[\cong]{i_*} & H_0(D^k) & \longrightarrow & H_0(D^k, S^{k-1}) \to 0 \\ & & \parallel & & \parallel & & \\ & & \mathbb{Z} & & \mathbb{Z} & & \end{array}$$

$$H_1(D^k, S^{k-1}) \cong im(\partial) = ker(i_*) = 0$$

$$\tilde{H}_n(S^1) \cong H_n(D^1, S^0) \cong \begin{cases} \mathbb{Z} & n=1 \\ 0 & n \neq 1 \end{cases}$$

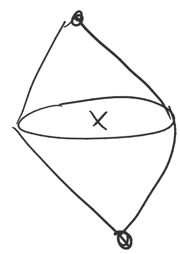
$\tilde{H}_n(S^1) \cong H_{n-1}(S^0) = 0 \quad n \neq 1$

$K>1$

Inductively $\quad \tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & n=k \\ 0 & n \neq k \end{cases}$ $\qquad \underbrace{\tilde{H}_n(S^k) \cong \tilde{H}_{n-1}(S^{k-1})}$

<u>Suspension</u>: Given any X

$$\Sigma X = X \times [0,1] \Big/ \begin{smallmatrix} (x,0) \sim (x',0) \\ (x,1) \sim (x',1) \end{smallmatrix}$$

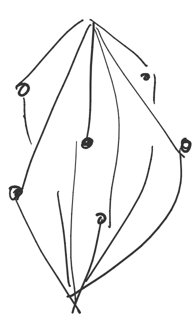


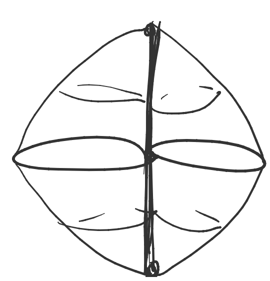
$$\Sigma X \cong CX \big/ X \times \{0\}$$

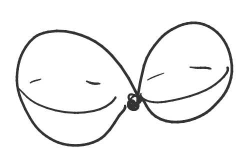
$\tilde{H}_n(\Sigma X) \cong H_n(CX, X) \qquad \tilde{H}_n(CX) = 0$

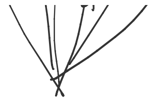
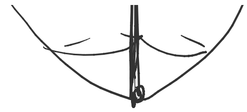
$$0 \to H_n(CX, X) \xrightarrow{\cong} H_{n-1}(X) \to 0$$

Take $X = \bigvee_j S^k \qquad \Sigma X \simeq \bigvee_j S^{k+1}$








Next goal: Simplicial homology $\cong$ Singular homology

Thm: Given X, + Δ-cx structure on X

$$H_n^{\Delta}(X) \cong H_n(X) \quad \forall n.$$

Proof: Given Δ-cx str, let X^K denote the K-skeleton: union of images of all simplex maps in Δ-cx of dim $\leq K$
$\sigma : \Delta^j \to X$

We will show by induction on K that

$$H_n^{\Delta}(X^k) \cong H_n(X^k)$$

If Δ-cx str has fin many dims with simplices then this will suffice.

$X^{k-1} \subset X^k$

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}^{\Delta}(X^k, X^{k-1}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_n^{\Delta}(X^{k-1}) \arrow[r, "i_*"] \arrow[d, "\cong"] & H_n^{\Delta}(X^k) \arrow[r, "j_*"] \arrow[d, "?"] & H_n^{\Delta}(X^k, X^{k-1}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{n-1}^{\Delta}(X^{k-1}) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X^k, X^{k-1}) \arrow[r, "\partial"] & H_n(X^{k-1}) \arrow[r, "i_*"] & H_n(X^k) \arrow[r, "j_*"] & H_n(X^k, X^{k-1}) \arrow[r, "\partial"] & H_{n-1}(X^{k-1}) \arrow[r] & \cdots
\end{tikzcd}$$

By induction on K assume $H_n^{\Delta}(X^{k-1}) \to H_n(X^{k-1})$ is an isomorphism

Base case
$$H_0^{\Delta}(X^0) \xrightarrow{\cong} H_0(X^0) \qquad H_n^{\Delta}(X^0) \xrightarrow{\cong} H_n(X^0)$$
$$H_0^{\Delta}(X^0) \cong \mathbb{Z}^{|X^0|}, \quad H_0(X^0) \cong \mathbb{Z}^{|X^0|}, \quad H_n^{\Delta}(X^0) = 0, \quad H_n(X^0) = 0$$

Claim: $H_n^{\Delta}(X^k, X^{k-1}) \longrightarrow H_n(X^k, X^{k-1})$ is an isomorphism $\forall n$

By five lemma: $H_n^{\Delta}(X^k) \to H_n(X^k)$ is an isomorphism.

By five lemma: $H_n^{\Delta}(X^k) \to H_n(X^k)$ is an isomorphism.

Proof: $H_n^{\Delta}(X^k, X^{k-1})$ is homology of

$$C_n^{\Delta}(X^k) / C_n^{\Delta}(X^{k-1}) = \begin{cases} 0 & \text{if } n<K \text{ or } n>K \\ \mathbb{Z}^{\#k\text{-simplices}} & n=K \end{cases}$$

$$\to 0 \xrightarrow{d} 0 \xrightarrow{d} \mathbb{Z}^{\#k\text{-simplices}} \xrightarrow{d} 0 \xrightarrow{d} 0 \xrightarrow{d} \cdots$$

$d=0$

$$\Rightarrow H_n^{\Delta}(X^k, X^{k-1}) \cong \begin{cases} 0 & \text{if } n \neq K \\ \mathbb{Z}^{\#k\text{simplices}} & \text{if } n=k \end{cases}$$

Next time, understand $\underline{H_n(X^k, X^{k-1})}$ using excision

Then extend to ∞ dimensional Δ-complexes.