Lecture 14

Monday, February 8, 2021 2:04 PM

Simplicial $\cong$ Singular

X^K - K-skeleton

$$\begin{tikzcd}
\cdots \arrow[r] & H^{\Delta}_{n+1}(X^k,X^{k-1}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H^{\Delta}_n(X^{k-1}) \arrow[r, "i_*"] \arrow[d, "\cong"] & H^{\Delta}_n(X^k) \arrow[r, "j_*"] \arrow[d, "?"] & H^{\Delta}_n(X^k,X^{k-1}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H^{\Delta}_{n-1}(X^{k-1}) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X^k,X^{k-1}) \arrow[r, "\partial"] & H_n(X^{k-1}) \arrow[r, "i_*"] & H_n(X^k) \arrow[r, "j_*"] & H_n(X^k,X^{k-1}) \arrow[r, "\partial"] & H_{n-1}(X^{k-1}) \arrow[r] & \cdots
\end{tikzcd}$$

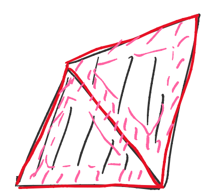
By inductive hypothesis blue isomorphisms

Claim: $H^{\Delta}_n(X^k, X^{k-1}) \to H_n(X^k, X^{k-1})$ is an isomorphism.

$$H^{\Delta}_n(X^k,X^{k-1}) \cong \begin{cases} \mathbb{Z}\langle k\text{-simplices}\rangle & n=k \\ 0 & \text{else} \end{cases}$$

$H_n(X^k, X^{k-1})$

X^{k-1} has a nbhd U in X^k which def retracts to X^{k-1}



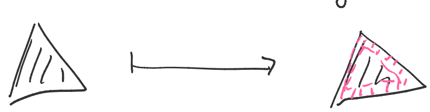
$$(*) \quad H_n(X^k,X^{k-1}) \underset{\substack{\uparrow \\ \text{homotopy equiv}}}{\cong} H_n(X^k,U) \underset{\substack{\uparrow \\ \text{excision}}}{\cong} H_n(X^k-X^{k-1},U-X^{k-1}) \underset{\substack{\uparrow \\ \text{homotopy equiv}}}{\cong} H_n(\sqcup D^k, \sqcup \partial D^k) \cong \mathbb{Z}\langle k\text{simplices}\rangle$$

$X^k - X^{k-1} \cong \sqcup \mathring{\Delta}^k_{\mathbb{R}}$

$U - X^{k-1} \leftarrow$ a nbhd of boundary in each

$H^{\Delta}_n(X^k,X^{k-1}) \longrightarrow H_n(X^k,X^{k-1}) \leftarrow$ is an isomorphism.

generators $\longmapsto$ generators



Alternate iso to $(*)$

$H_n(X^k,X^{k-1}) \cong \tilde{H}_n(X^k/X^{k-1})$

$X^k/X^{k-1} \cong \vee S^k$

Five lemma $\Rightarrow H^{\Delta}_n(X^k) \to H_n(X^k)$ is an isomorphism.

$H_n(D^k, \partial D^k) \cong \mathbb{Z}$ generator $[D^k]$

$H_n(D^n, \partial D^n) \cong \mathbb{Z}$ generator $[D]$

__Idea for infinite dim'l case:__

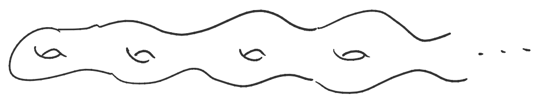
$$i_*^k : H_n(X^k) \longrightarrow H_n(X)$$

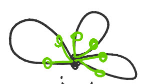
For any $\alpha \in H_n(X)$, it will be in image of i_*^k for suff large k.

__Proof:__

__Observation 1:__ Any compact subset of X will intersect only finitely many interiors of simplices in a fixed Δ-complex.

coming from topology property of Δ complexes (3)





__Observation 2:__ Any singular chain is represented by a finite integer combination of maps $\sigma_i : \Delta^l \to X$ with compact image.

$\Rightarrow$ Any sing chain has image in X^k for some large enough k.

$$\beta = [\textstyle\sum_i m_i \sigma_i] \in H_l(X^k) \xrightarrow{i_*} H_l(X) \ni \alpha$$

$$\begin{tikzcd}
\gamma \arrow[rr, mapsto] & & i_*(\gamma) \\
H_l^\Delta(X^k) \arrow[r, "i_*"] \arrow[d, "\cong"'] & H_l^\Delta(X) \arrow[d, "f"] & \\
H_l(X^k) \arrow[r, "i_*"] & H_l(X) & \\
\beta \arrow[rr, mapsto] & & \alpha
\end{tikzcd}$$

f is surjective

__Injectivity:__

$$\begin{tikzcd}
0 = [\sum m_j \sigma_j] \in H_l^\Delta(X^k) \arrow[r, "i_*"] \arrow[d, "\cong"] & H_l^\Delta(X) \ni \gamma = [\sum m_j \sigma_j] = 0 \arrow[d] \\
H_l(X^k) \arrow[r] & H_l(X)
\end{tikzcd}$$

$[\sum m_j \sigma_j] = 0$

$f(\gamma) = 0 \leftarrow$ assume

$= [\sum m_j \sigma_j]$

$\downarrow$ $H_\ell(\ldots) \longrightarrow H_\ell(X)$ $\quad f(\gamma) = 0 \leftarrow$ assu...

$[\Sigma m_j \sigma_j] = 0 \longrightarrow [\Sigma m_j \sigma_i]$

$$\Sigma m_j \sigma_j = d_{\ell+1}(\Sigma n_i \tau_i) \qquad \tau_i : \Delta^{\ell+1} \to X$$

has image in some X^k

$\boxtimes$

Singular homology : good b/c we could show it was homotopy invt + excision, ...

Simplicial homology: good b/c computable

Still hard to calculate b/c need lots of K-simplices in Δ-complex

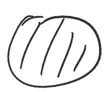
Next goal: Define another homology theory, easier to calculate in terms of # of generators

Cellular homology $\leftarrow$ defined for any cell complex

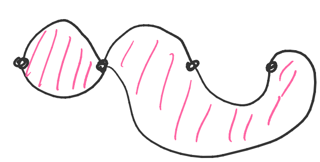
We will show it is isomorphic to singular homology.

Harder thing about cellular homology -- harder to calculate d_n maps

Cell D^n



Cell complex: Start with 0-cells D^0 glue on 1-cells along boundary



Glue K-cell to (K-1)-skeleton by any continuous map

$$f : \partial D^k \longrightarrow X^{k-1}$$
$$\partial D^k = S^{k-1}$$

Degree:

Recall $\quad H_K(S^n) \cong \begin{cases} \mathbb{Z} & K = 0, n \\ 0 & \text{else} \end{cases}$

Given a continuous map $f: S^n \to S^n$

get $f_*: H_n(S^n) \to H_n(S^n)$ choose generator $\mathbb{1}$ of $H_n(S^n)$

$$\begin{array}{ccc} H_n(S^n) & \to & H_n(S^n) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

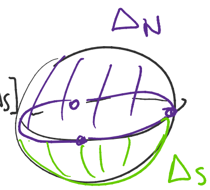
f_* is determined by $f_*(\mathbb{1}) = d\,\mathbb{1}$ for some $d \in \mathbb{Z}$

$$f_*(m\mathbb{1}) = dm\mathbb{1}$$

Call d the <u>degree</u> of f.

<u>Example</u>:

$H_n(S^n) = \mathbb{Z}[\Delta_N - \Delta_S]$



$[\Delta_N - \Delta_S] = \mathbb{1}$

$(-1)\mathbb{1} = [\Delta_S - \Delta_N]$

- $id: S^n \to S^n$ $\deg(id) = 1$
- reflection of last coordinate: $S^n \to S^n$ $\deg(\text{reflection}) = -1$
- $\deg(f \circ g) = \deg(f) \cdot \deg(g)$

  $(f \circ g)_* = f_* \circ g_*$
- constant map $c: S^n \to S^n$

n-dim'l manifolds

More generally: $f: X^n \to Y^n$ $\deg(f)$ defined similarly

$$\begin{array}{ccc} H_n(X^n) & \to & H_n(Y^n) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$