Lecture 15

Wednesday, February 10, 2021   1:56 PM

HW 4 due Friday

Degree $f: S^n \to S^n$

$$f_*: \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \to \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \qquad f_*(\alpha) = d\alpha$$

$$\deg(f) = d.$$

* $\deg(id) = 1$
* $\deg(\text{reflection}) = -1$
* $\deg(\text{constant}) = 0.$

For any $f: S^n \to S^n$ which is not surjective, $\deg(f) = 0$

Pf:

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[dr, "\tilde{f}"'] & & S^n \\ & S^n - \{pt\} \arrow[ur, "i"'] & \end{tikzcd} \qquad \begin{tikzcd} H_n(S^n) \arrow[rr, "f_*"] \arrow[dr, "\tilde{f}_*"'] & & H_n(S^n) \\ & H_n(S^n - \{pt\}) \arrow[ur, "i_*"'] & \end{tikzcd} \Rightarrow f_* = 0.$$

$$H_n(S^n - \{pt\}) = 0$$

More generally, suppose $f: S^n \to S^n$ and $y \in S^n$ s.t.

$$f^{-1}(y) = \{x_1, \dots, x_k\} \quad (\text{finite})$$

$$\deg(f) = \sum_{i=1}^{k} \deg_{x_i} f$$

← local degree

To define $\deg_{x_i} f$: choose a nbhd U_i of x_i mapping into a nbhd V of y.

make $U_1, U_2, \dots, U_k$ disjoint

$$f_*: \underset{\mathbb{Z}}{\underset{\cong}{H_n(U_i, U_i - \{x_i\})}} \longrightarrow \underset{\mathbb{Z}}{\underset{\cong}{H_n(V, V - \{y\})}} \qquad f_*(1) = d_i \cdot 1$$

$\sigma: \Delta^n \to U_i$

$$\deg_{x_i}(f) := d_i$$

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Claim: $\deg f = \sum_{i=1}^{k} \deg_{x_i}(f)$

Proof:

local degree at x_i

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \cong \mathbb{Z} \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d] & H_n(V, V-\{y\}) \cong \mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z} \cong H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - \{x_1, \dots, x_k\}) \arrow[l] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \cong \mathbb{Z} \\
 & \mathbb{Z} \cong H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j_*"] \arrow[r, "f_*"] & H_n(S^n) \cong \mathbb{Z} \arrow[u, "j_*"]
\end{tikzcd}$$

isomorphism by excision

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isomorphism by exact seq of pair

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$$\underset{= 0}{H_n(S^n - \{y\})} \xrightarrow{i_*} H_n(S^n) \xrightarrow{j_*} H_n(S^n, S^n - \{y\}) \to \underset{= 0}{H_{n-1}(S^n - \{y\})}$$

$\deg(f)$

$$H_n(S^n, S^n - \{x_1, \dots, x_k\}) \cong \mathbb{Z}^k$$

$$\cong H_n\left(\bigvee_k S^n\right)$$

quotient of $S^n / S^n - (D_1 \cup \dots \cup D_k)$

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \cong \mathbb{Z} \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "q_i"] & H_n(V, V-\{y\}) \cong \mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z} \cong H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - \{x_1, \dots, x_k\}) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \cong \mathbb{Z} \\
 & \mathbb{Z} \cong H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j_*"] \arrow[r, "f_*"] & H_n(S^n) \cong \mathbb{Z} \arrow[u, "j_*"]
\end{tikzcd}$$

$1 \mapsto \deg_{x_i} f$

$(0, \dots, 1, \dots, 0) \mapsto \deg_{x_i} f$

$(1, \dots, 1) = \sum_{i=1}^{k} (0, \dots, 1, \dots, 0) \mapsto \deg f$

i^{th} pos

$1 \mapsto 1$

$1 \mapsto (1, \dots, 1)$

$1 \mapsto \deg(f)$

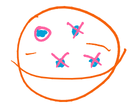
$$f_*(0, \dots, 1, \dots, 0) = \deg_{x_i} f$$

$$\sum_{i=1}^{k} f_*(0, \dots, 1, \dots, 0) = \deg f$$

i^{th} position

$p_i : H_n(S^n, S^n - \{x_1, \dots, x_n\}) \to H_n(S^n, S^n - \{x_i\})$

$\quad\quad (a_1, \dots, a_n) \longmapsto a_i$



$\ell_i : H_n(U_i, U_i - \{x_i\}) \to H_n(S^n, S^n - \{x_1, \dots, x_n\})$

$\quad 1 \longmapsto (0, \dots, 1, \dots 0)$

$\quad\quad\quad\quad \uparrow$ ith position

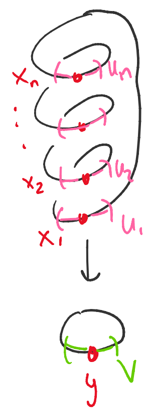
$j_* : H_n(S^n) \to H_n(S^n, S^n - \{x_1, \dots, x_n\})$

$\quad 1 \longmapsto (1, 1, \dots, 1)$

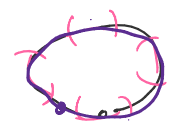
Example:

$f : S^1 \to S^1$

$e^{i\theta} \mapsto e^{in\theta}$



$f^{-1}(1) = \{ e^{2\pi i k/n} \mid 0 \le k \le n-1 \}$



$\deg(f) = \sum_{i=1}^{n} \deg_{x_i} f$

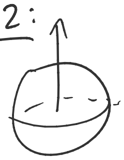
$\deg_{x_i} f \quad\quad f_* : H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\})$

$f : U_i \to V$ is a homeomorphism

$\deg_{x_i} f = \pm 1$

In this case $\deg(f) = \sum_{i=1}^{n} 1 = n$

Example 2:

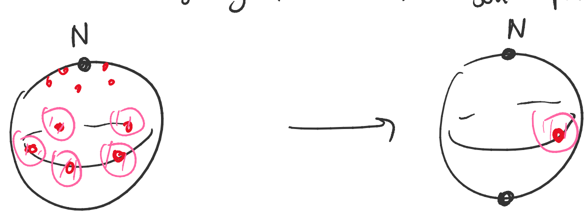


cylindrical words (r, θ, z)

$f : S^2 \to S^2$

$(r, \theta, z) \mapsto (r, n\theta, z)$

If $y \neq$ a north/south pole $\quad f^{-1}(y) = \{x_1, \dots, x_n\}$



$\deg f = n$

$\deg_N f = \deg f = n$

For $x \neq N, S$

for $x \neq N, S$

$\deg_x f = 1$

$\deg_N f = n \qquad \deg_S f = n$

$$\begin{array}{ccc} f: \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^1 \\ & & \rotatebox{90}{$\in$} \\ f^{-1}(\infty) & & \infty \end{array}$$