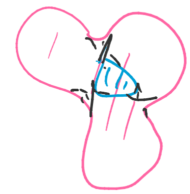
Lecture 16

Friday, February 12, 2021 2:06 PM

Monday is a holiday, Turn in HW4

Compute w/ excision (S^3, N) $\longrightarrow H_k(S^3 - N, \partial N)$

$$\cdots \to H_k(\partial N) \to \underbrace{H_k(S^3 - N)}_{?} \to H_k(S^3 - N, \partial N) \to \cdots$$



$H_2(S^3 - N, \partial N)$ generated by

$$H_2(S^3 - N, \partial N) \xrightarrow{\partial} H_1(\partial N) \to H_1(S^3 - N) \to$$

[Seif surface] $\longrightarrow$ [boundary of Seif surface in the tube]

Cellular homology:

Given a cell complex X

X^k – k-skeleton (union of the l-cell for $l \le k$)

Want a chain complex $C_n^{cw}(X) = \mathbb{Z}\langle n\text{-cells}\rangle$

Need to define: d_n^{cw}, want it to capture gluing information of how each n-cell is glued to X^{n-1}.

Observe:

$$H_n(X^k, X^{k-1}) = \begin{cases} \mathbb{Z}\langle k\text{-cells}\rangle & n = k \\ 0 & n \ne k \end{cases}$$

$$\left(X^k / X^{k-1} \cong \bigvee S^k \right)$$

Set $C_n^{cw}(X) := H_n(X^n, X^{n-1}) \cong \mathbb{Z}\langle n\text{-cells}\rangle$

To define d_n^{cw}:

$$H_n(X^n) \to \cdots$$

$$\begin{tikzcd}[column sep=small]
 & & H_n(X^n) \arrow[dr, "j_*"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[rr, "d_{n+1}^{cw}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n^{cw}"] \arrow[dr, "\partial"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
\cdots \arrow[ur] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_*"] \arrow[dr] & & \\
 & & & \cdots \arrow[ur] & & \cdots &
\end{tikzcd}$$

$d_n^{cw} := j_* \circ \partial$

$d_n^{cw} \circ d_{n+1}^{cw} = (j_* \circ \partial) \circ (j_* \circ \partial) = j_* \circ (\underbrace{\partial \circ j_*}_{0}) \circ \partial = 0$

<u>Need</u>: $d_n^{cw} \circ d_{n+1}^{cw} = 0$ ✓

$(C_n^{cw}(X), d_n^{cw})$ is a chain complex. So we get homology

$$H_n^{cw}(X) := \operatorname{Ker} d_n^{cw} / \operatorname{im} d_{n+1}^{cw}.$$

<u>Goals</u>: ① How to compute d_n^{cw}. $\qquad H_n(X^n, X^{n-1}) \cong \mathbb{Z}\langle n\text{-cells}\rangle$

(In terms of gluing maps + degrees)

② Show $H_n^{cw}(X) \cong H_n(X)$ ← follow mostly from exact sequences

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①

$$\begin{tikzcd}
{[e_\alpha] \in H_n(X^n, X^{n-1})} \arrow[rr, "d_n^{cw}"] \arrow[dr, "\partial"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_{n-1}(X^{n-1}) \arrow[ur, "j_*"'] &
\end{tikzcd}$$

Let e_α be an n-cell of X $\qquad e_\alpha : D^n \to X^n \subseteq X$ $\qquad e_\alpha|_{\mathring{D}^n}$ is a homeomorphism to its image

$e_\alpha|_{S^{n-1}} : S^{n-1} \to X^{n-1}$

$d_n^{cw}([e_\alpha]) = j_* \circ \partial([e_\alpha])$

$= j_*[e_\alpha|_{S^{n-1}}]$

$= [e_\alpha|_{S^{n-1}}] \in H_{n-1}(X^{n-1}, X^{n-2})$

$$\begin{tikzcd}
 & & C_{n-1}(X^{n-1}) \arrow[d] & \ni e_\alpha|_{S^{n-1}} \arrow[d] \\
e_\alpha \arrow[d, mapsto] & C_n(X^n) \arrow[r, "d_n"] \arrow[d] & C_{n-1}(X^n) & d_n e_\alpha = e_\alpha|_{S^{n-1}} \\
 & &
\end{tikzcd}$$

← (a sum of simplices)

simplices)

$$= [e_\alpha|_{S^{n-1}}] \in H_{n-1}(X^{n-1}, X^{n-2})$$

gluing map

$$e_\alpha \in C_n(X) \quad \Big\downarrow \qquad [e_\alpha] \in C_n(X^n)/C_n(X^{n-1})$$

$$d_n([e_\alpha]) = \underline{[e_\alpha|_{S^{n-1}}]} \in H_{n-1}(X^{n-1}, X^{n-2}) \cong \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \xrightarrow{p^\gamma_*} H_{n-1}(S^{n-1})$$

$$X^{n-1}/X^{n-2} \cong \bigvee S^{n-1}$$ ← one for each $n-1$ cell

$$p^\gamma : X^{n-1}/X^{n-2} \longrightarrow S^{n-1}$$

↑ quotient everything except the γ^{th} $(n-1)$ cell to a point.

$$\begin{array}{ccc} H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{p^\gamma_*} & H_{n-1}(S^{n-1}) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}\langle (n-1)\text{-cells}\rangle & & \mathbb{Z} \\ \sum_\beta n_\beta f_\beta & \longmapsto & n_\gamma \end{array}$$

f_β ← $(n-1)$-cell

I can learn the coefficients of an elt of $H_{n-1}(X^{n-1}, X^{n-2})$ by knowing values of its image under p^γ_* for each γ indexing $(n-1)$ cells.

$$d^{CW}_n([e_\alpha]) = [e_\alpha|_{S^{n-1}}] \in H_{n-1}(X^{n-1}, X^{n-2})$$

↗ probe this elt with p^γ_*

knowing $p^\gamma_*([e_\alpha|_{S^{n-1}}])$ for all γ, tells us what $[e_\alpha|_{S^{n-1}}]$ is.

$$S^{n-1} \xrightarrow{e_\alpha|_{S^{n-1}}} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow{p^\gamma} S^{n-1}$$

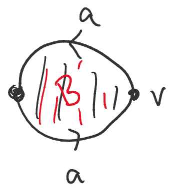
$$S^{n-1} = \partial D^n$$

$$S^{n-1} \xrightarrow{p^\gamma \circ q \circ e_\alpha|_{S^{n-1}}} S^{n-1}$$ this map has a degree an integer

$$H_{n-1}(S^{n-1}) \xrightarrow{(p^\gamma \circ q \circ e_\alpha|_{S^{n-1}})_*} H_{n-1}(S^{n-1})$$ (encodes degree of gluing map)

$$p_*^{\gamma}\left(d_n^{CW}[e_\alpha]\right) = p_*^{\gamma}\left(\left[e_\alpha|_{S^{n-1}}\right]\right) = p_*^{\gamma} \circ q_* \circ (e_\alpha|_{S^{n-1}})_*(1) = \text{degree}\left(p^{\gamma} \circ q \circ e_\alpha|_{S^{n-1}}\right)$$

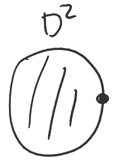
$\mathbb{RP}^2$

$X =$ 

$$\cdots \to 0 \to \mathbb{Z}\langle B\rangle \xrightarrow{d_2^{CW}} \mathbb{Z}\langle a\rangle \xrightarrow{d_1^{CW}} \mathbb{Z}\langle v\rangle \to 0$$

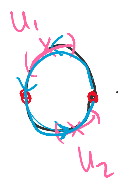
$d_2(B) = 2a$ $\qquad a \longmapsto 0$

$$H_1 = \ker d_1 / \operatorname{im} d_2 = \mathbb{Z}\langle a\rangle / \mathbb{Z}\langle 2a\rangle \cong \mathbb{Z}/2$$



$\xrightarrow{e_B} X$

$\deg(e_B|_{S^1}) = 1 + 1 = 2$



$\xrightarrow{e_B|_{S^1}} X' =$