

Lecture 17

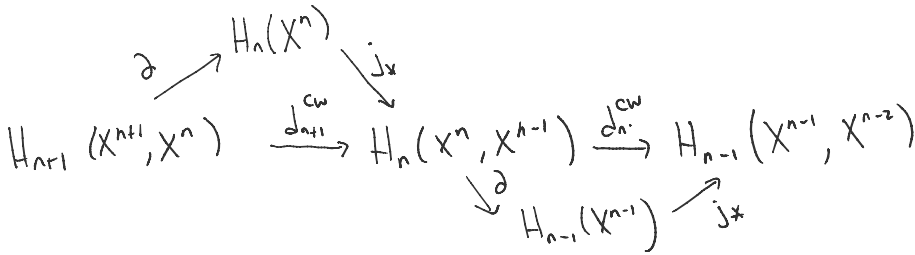
Wednesday, February 17, 2021 2:04 PM

Cellular homology

X cell complex

$$C_n^{CW}(X) = \mathbb{Z}\langle n\text{-cells} \rangle = H_n(X^n, X^{n-1})$$

$$d_n^{CW} = j_* \circ \partial$$



Interpreted $d_n^{CW} : H_n(X^n, X^{n-1}) \rightarrow H_{n-1}(X^{n-1}, X^{n-2})$

$$\cong \tilde{H}_{n-1}(X^{n-1}/X^{n-2})$$

$$X^{n-1}/X^{n-2} \cong V S^{n-1}$$

$$\downarrow q_*^i$$

$$q^i : V S^{n-1} \rightarrow S^{n-1}$$

quotients out all but i^{th} S^{n-1} to a point

$$q_*^i : \mathbb{Z}\langle (n-1)\text{-cells} \rangle \rightarrow \mathbb{Z}\langle i^{\text{th}} (n-1)\text{cell} \rangle$$

$$q_*^i \circ d_n^{CW}(e_n^\alpha) = \deg(e_n^\alpha|_{S^{n-1}} : S^{n-1} \rightarrow X^{n-1} \rightarrow X^{n-1}/X^{n-2} \xrightarrow{q^i} S^{n-1})$$

$$d_n^{CW}(e_n^\alpha) = (q_*^1 \circ d_n^{CW}(e_n^\alpha), \dots, q_*^N \circ d_n^{CW}(e_n^\alpha))$$

Goal: $H_n^{CW}(X) = \text{Ker } d_n^{CW} / \text{im } d_{n+1}^{CW}$ is isomorphic to $H_n(X)$.

Lemma 1: $H_n(X^k) = 0$ if $n > k$.

Proof: By exact sequence

$$\dots \rightarrow H_{n+1}(X^k, X^{k-1}) \rightarrow H_n(X^{k-1}) \xrightarrow{j_*} H_n(X^k) \rightarrow H_n(X^k, X^{k-1}) \rightarrow \dots$$

whenever $n > k$

$$H_n(X^k, X^{k-1}) = H_{n+1}(X^k, X^{k-1}) = 0$$

$$\Rightarrow H_n(X^{k+1}) \cong H_n(X^k)$$

$$H_n(X^k) \cong H_n(X^{k-1}) \cong H_n(X^{k-2}) \cong \dots \cong H_n(X^0) = 0 \quad \square$$

Lemma 2: $H_n(X^{n+1}) \cong H_n(X)$ (induced by inclusion)

pf: $H_{n+1}(X^{k+1}, X^k) \rightarrow H_n(X^k) \xrightarrow{i_*} H_n(X^{k+1}) \rightarrow H_n(X^{k+1}, X^k)$

For any $k > n$ 

$$H_n(X^k) \xrightarrow{\cong} H_n(X^{k+1})$$

Applies for $k=n+1$ + higher:

$$H_n(X^{n+1}) \cong H_n(X^{n+2}) \cong H_n(X^{n+3}) \cong \dots \cong H_n(X^{n+N})$$

If X is finite dim all complex $X = X^{n+N}$ so done.

If X is infinite dim, then make argument similar to case in simplicial = singular.

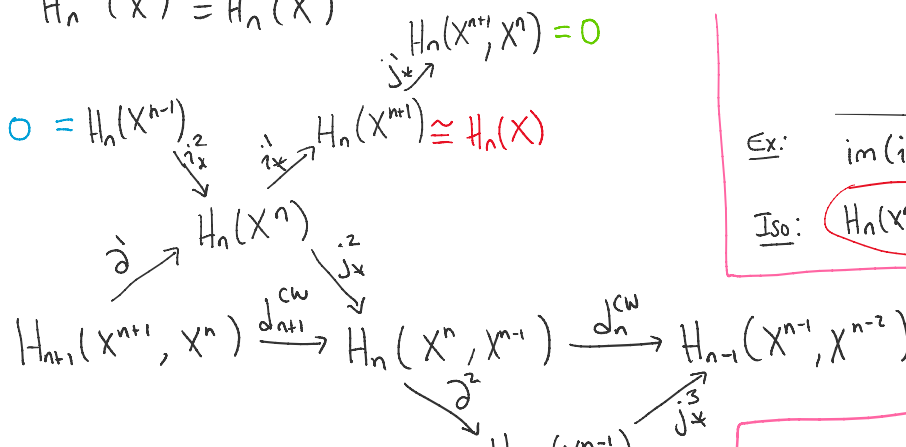
For any class in $H_n(X)$, it is represented by a class in

$$H_n(X^M) \text{ for } M \text{ large (images of simplices are compact)}$$

Using that homology cycles + chains are compact + thus contained in some X^M . \square

Lemmas: $H_n(X^k) = 0$ if $n > k$ $H_n(X^{n+1}) \cong H_n(X)$

Theorem: $H_n^{CW}(X) \cong H_n(X)$



Want to show

$$\text{Ker } d_n^{CW} / \text{im } d_{n+1}^{CW} \cong H_n(X)$$

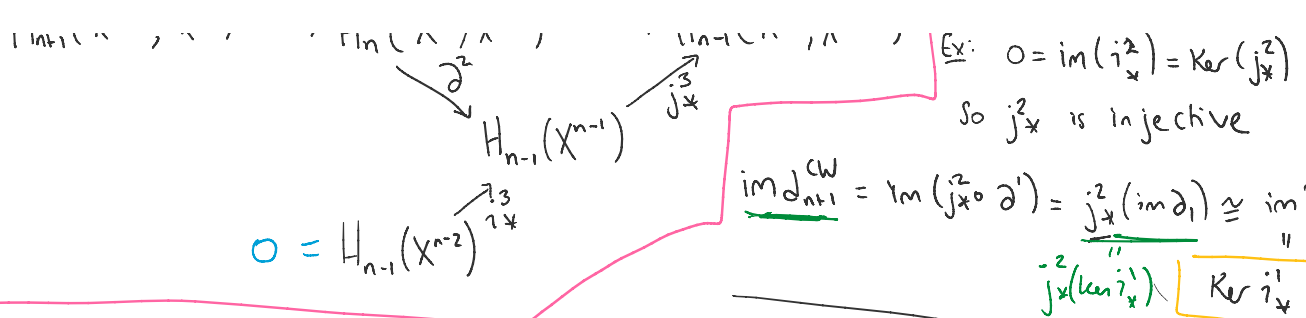
Ex: $\text{im}(i'_*) = \text{Ker}(j'_*) = H_n(X^{n+1}) \cong H_n(X)$

Iso: $H_n(X^n) / \text{Ker } i'_* \cong \text{im}(i'_*) \cong H_n(X)$

Ex: $\text{Ker } i'_* = \text{im } \partial'$

Ex: $0 = \text{im}(i_x^2) = \text{Ker}(j_x^2)$

So i_x^2 is injective



$\text{Ker}(d_n^{CW}) = \text{Ker}(j_*^3 \circ \partial^2)$

Exactness: j_*^3 is injective

$\text{Ker}(d_n^{CW}) = \text{Ker}(\partial^2)$

($j_*^3 \circ \partial^2(a) = 0$ then $\partial^2(a) = 0$)

Exactness: $\text{Ker}(\partial^2) = \text{im}(j_*^2) \cong H_n(X^n)$

because j_*^2 is injective

j_*^2 induces an isomorphism taking $H_n(X^n) / \text{Ker}(i_*^1) \cong H_n(X)$

