Lecture 17

Wednesday, February 17, 2021 2:04 PM

Cellular homology

X cell complex

$C_n^{CW}(X) = \mathbb{Z}\langle n\text{-cells}\rangle = H_n(X^n, X^{n-1})$

$d_n^{CW} = j_* \circ \partial$

$$\begin{tikzcd}
 & H_n(X^n) \arrow[dr, "j_*"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[rr, "d_{n+1}^{CW}"] & & H_n(X^n, X^{n-1}) \arrow[dr, "\partial"] \arrow[rr, "d_n^{CW}"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_*"] &
\end{tikzcd}$$

Interpreted $d_n^{CW} : H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$

$$H_{n-1}(X^{n-1}, X^{n-2}) \cong \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \xrightarrow{q_*^i} \tilde{H}_{n-1}(S^{n-1})$$

$X^{n-1}/X^{n-2} \overset{\text{homeo}}{\cong} \bigvee S^{n-1}$

$q^i : \bigvee S^{n-1} \longrightarrow S^{n-1}$

quotients out all but i^{th} S^{n-1} to a point

$q_*^i : \mathbb{Z}\langle (n-1)\text{-cells}\rangle \rightarrow \mathbb{Z}\langle i^{th}\ (n-1)\text{cell}\rangle$

$$q_*^i \circ d_n^{CW}(e_n^\alpha) = \deg\left(e_n^\alpha\big|_{S^{n-1}} : S^{n-1} \longrightarrow X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \xrightarrow{q^i} S^{n-1}\right)$$

$$d_n^{CW}(e_n^\alpha) = \left(q_*^1 \circ d_n^{CW}(e_n^\alpha), \dots, q_*^N \circ d_n^{CW}(e_n^\alpha)\right)$$

Goal: $H_n^{CW}(X) = \ker d_n^{CW} / \operatorname{im} d_{n+1}^{CW}$ is isomorphic to $H_n(X)$.

Lemma 1: $H_n(X^k) = 0$ if $n > k$.

Proof: By exact sequence

$$\dots \rightarrow \underset{\overset{\shortparallel}{0}}{H_{n+1}(X^k, X^{k-1})} \longrightarrow H_n(X^{k-1}) \xrightarrow{i_*} H_n(X^k) \rightarrow \underset{\overset{\shortparallel}{0}}{H_n(X^k, X^{k-1})} \rightarrow \dots$$

whenever $n > k$

$H_n(X^k, X^{k-1}) = H_{n+1}(X^k, X^{k-1}) = 0$

$\Rightarrow H_n(X^{k-1}) \cong H_n(X^k)$

$$H_n(X^k) \cong H_n(X^{k-1}) \cong H_n(X^{k-2}) \cong \dots \cong H_n(X^0) = 0 \quad \boxtimes$$

Lemma 2: $H_n(X^{n+1}) \cong H_n(X)$ (induced by inclusion)

Pf:
$$\underset{\overset{\shortparallel}{0}}{H_{n+1}(X^{k+1}, X^k)} \to H_n(X^k) \xrightarrow{i_*} H_n(X^{k+1}) \to \underset{\overset{\shortparallel}{0}}{H_n(X^{k+1}, X^k)}$$

For any $k>n$

$$H_n(X^k) \xrightarrow{\cong} H_n(X^{k+1})$$

Applies for $k=n+1$ + higher:
$$H_n(X^{n+1}) \cong H_n(X^{n+2}) \cong H_n(X^{n+3}) \cong \dots \cong H_n(X^{n+N})$$

If X is finite dim'l complex $X = X^{n+N}$ so done.

If X is infinite dim'l, then make argument similar to case in simplicial = singular.

For any class in $H_n(X)$, it is represented by a class in

$H_n(X^M)$ for M large (images of simplices are compact)

Using that homology cycles + chains are compact + thus contained in some X^M. $\boxtimes$

[**Lemmas:** $H_n(X^k) = 0$ if $n > k$ $\qquad H_n(X^{n+1}) \cong H_n(X)$]

Theorem: $H_n^{CW}(X) \cong H_n(X)$

$$\begin{tikzcd}
 & & & H_n(X^{n+1}, X^n) = 0 & \\
0 = H_n(X^{n-1}) \arrow[dr, "i_*^2"] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur, "j_*"] & & \\
 & H_n(X^n) \arrow[ur, "i_*^1"] \arrow[dr, "j_*^2"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial^1"] \arrow[rr, "d_{n+1}^{CW}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n^{CW}"] \arrow[dr, "\partial^2"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_*^3"] &
\end{tikzcd}$$

Want to show
$$\operatorname{Ker} d_n^{CW} / \operatorname{im} d_{n+1}^{CW} \cong H_n(X)$$

Ex: $\operatorname{im}(i_*^1) = \operatorname{Ker}(j_*^1) = H_n(X^{n+1}) \cong H_n(X)$

Iso: $H_n(X^n) / \boxed{\operatorname{Ker} i_*^1} \cong \operatorname{im}(i_*^1) \cong H_n(X)$

Ex: $\operatorname{Ker} i_*^1 = \operatorname{im} \partial^1$

Ex: $0 = \operatorname{im}(i_*^2) = \operatorname{Ker}(j_*^2)$

So i_*^2 is injective

$H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$

$$\begin{array}{c} \partial^2 \searrow \qquad \nearrow j^3_* \\ H_{n-1}(X^{n-1}) \\ \nearrow i^3_* \\ 0 = H_{n-1}(X^{n-2}) \end{array}$$

Ex: $0 = \operatorname{im}(i^2_*) = \ker(j^2_*)$

So j^2_* is injective

$\operatorname{im} d^{CW}_{n+1} = \operatorname{im}(j^2_* \circ \partial^1) = j^2_*(\operatorname{im} \partial_1) \cong \operatorname{im} \partial_1$

$j^2_*(\ker i^1_*) = j^2_*(\operatorname{im}\partial_1)$, $\operatorname{im}\partial_1 = \boxed{\ker i^1_*}$

$\ker(d_n^{CW}) = \ker(j^3_* \circ \partial^2)$

Exactness: j^3_* is injective

$\ker(d_n^{CW}) = \ker(\partial^2)$

$(j^*_3 \circ \partial^2(a) = 0$ then $\partial^2(a) = 0)$

Exactness: $\ker(\partial^2) = \operatorname{im}(j^2_*) \cong H_n(X^n)$

$\uparrow$ because j^2_* is injective

j^2_* induces an isomorphism taking $H_n(X^n) / \ker(i^1_*) \longrightarrow j^2_*(H_n(X^n)) / j^2_*(\ker(i^1_*))$

$j^2_*(H_n(X^n)) = \ker d_n^{CW}$

$j^2_*(\ker(i^1_*)) = \operatorname{im} d^{CW}_{n+1}$

$= H_n^{CW}(X)$

$H_n(X) \cong H_n(X^n) / \ker(i^1_*)$