Lecture 19

Monday, February 22, 2021 2:04 PM

Excision + exact sequence of pair ← good for cutting operations (relating homology of X and $A \subset X$ to homology of $X \setminus \mathring{A}$)

$$H_n(X,A) \cong H_n(\underline{X \setminus \mathring{A}}, \underline{\underline{\bar{A} \setminus \mathring{A}}})$$

For gluing, use <u>Mayer-Vietoris sequence</u>:

<u>Theorem</u>: Let $A, B \subset X$ s.t. $X = \mathring{A} \cup \mathring{B}$ then we have an (long) exact sequence

$$\longrightarrow H_n(A \cap B) \xrightarrow{f_n} H_n(A) \oplus H_n(B) \xrightarrow{g_n} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \longrightarrow \cdots$$

<u>Proof</u>: This long exact sequence is induced by a short exact sequence of chain complexes, together with the refinement lemma.

$\mathcal{U} = \{A, B\}$ $\quad C_n^{\mathcal{U}}(X)$ ← sums of chains in A with chains in B.

Recall that $H_n^{\mathcal{U}}(X) \cong H_n(X)$.

$$0 \to C_n(A \cap B) \xrightarrow{f_n^\circ} \underline{C_n(A)} \oplus \underline{C_n(B)} \xrightarrow{g_n^\circ} C_n^{\mathcal{U}}(X) \longrightarrow 0$$

$f_n^\circ(c) := (c, -c)$

$g_n^\circ(x, y) := \underline{x} + \underline{y}$

Exactness:

f_n° injective ✓

$\operatorname{im} f_n^\circ = \operatorname{Ker} g_n^\circ$ ✓

g_n° surjective ✓

Short exact sequence of chains ⇝ Long exact seq on homology

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to \underset{\displaystyle H_n(X)}{\overset{\displaystyle H_n^{\mathcal{U}}(X)}{\wr\|}} \to H_{n-1}(A \cap B) \to \cdots$$

∎

$$H_n(A\cap B) \xrightarrow{f_n} H_n(A)\oplus H_n(B)$$
$$[c] \longmapsto ([c], -[c])$$

$$H_n(A)\oplus H_n(B) \xrightarrow{g_n} H_n(X)$$
$$([x],[y]) \longmapsto [x]+[y]$$

<u>Connecting map:</u> $H_n(X) \cong \underline{H_n^u(X)} \xrightarrow{\partial_n} H_{n-1}(A\cap B)$

$[x+y]$

$\uparrow$ x is a chain on A

$\nwarrow$ y is a chain on B

$$\begin{tikzcd}
 & & d_n x \arrow[d, mapsto, "\circledast"'] & C_{n-1}(A\cap B) \arrow[d] \\
(x,y) & C_n(A)\oplus C_n(B) \arrow[d] \arrow[rr, "d_n"] & (d_nx, d_ny) & C_{n-1}(A)\oplus C_{n-1}(B) \\
{[x+y]} \arrow[uurrr, bend left] & C_n^u(X) & &
\end{tikzcd}$$

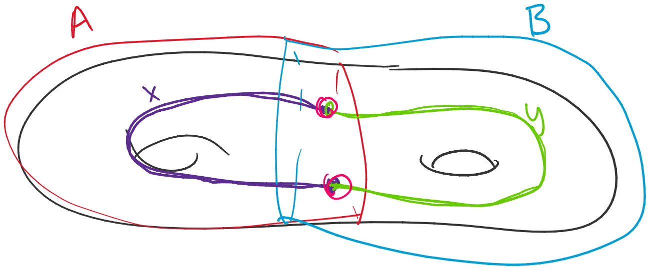
$$d_n(x+y) = 0$$

$$d_n x = -d_n y$$

$\uparrow$ in A $\qquad$ $\uparrow$ in B

$$d_n x = -d_n y \in C_{n-1}(A\cap B)$$

$\circledast$ $(d_nx, d_ny) = (d_nx, -d_nx)$



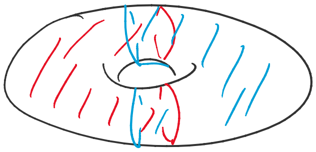
$$\partial([x+y]) = \begin{matrix} +\circ \\ -\circ \end{matrix} \in H_{n-1}(A\cap B)$$

<u>Application 1:</u> <u>Connected sums</u>

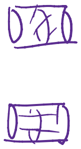
<u>Application 2:</u> <u>Dehn surgery</u>

<u>First:</u> Quick example in concrete case:

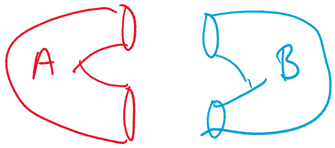
<u>Example:</u> $T^2 =$



$A\cap B =$



$\simeq S^1 \sqcup S^1$

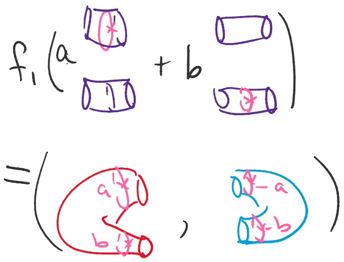


$A \simeq S^1$ $\qquad$ $B \simeq S^1$

$$\to \underset{\substack{\shortparallel \\ 0}}{H_2(A\cap B)} \to \underset{\substack{\shortparallel \quad \shortparallel \\ 0 \oplus 0}}{H_2(A)\oplus H_2(B)} \to H_2(X) \xrightarrow{\partial_2} \underset{\substack{\wr\| \\ \mathbb{Z}\oplus\mathbb{Z}}}{H_1(A\cap B)} \xrightarrow{f_1} \underset{\substack{\wr\| \quad \wr\| \\ \mathbb{Z} \oplus \mathbb{Z}}}{H_1(A)\oplus H_1(B)} \xrightarrow{g_1} \underline{H_1(X)}$$

$$\xrightarrow{\textcircled{∂_0}} \underset{\substack{\wr\| \\ \mathbb{Z}\oplus\mathbb{Z}}}{H_0(A\cap B)} \xrightarrow{\textcircled{f_0}} \underset{\substack{\wr\| \quad \wr\| \\ \mathbb{Z} \oplus \mathbb{Z}}}{H_0(A)\oplus H_0(B)} \xrightarrow{g_0} \underset{\substack{\wr\| \\ \mathbb{Z}}}{H_0(X)} \to 0$$

∂_2 injective $\Rightarrow$ $H_2(X) \cong \operatorname{im}\partial_2 = \operatorname{Ker} f_1 \cong \mathbb{Z}$



$\operatorname{im}(f_1) = \{(c,-c)\} = \operatorname{Ker} g_1$

$f_1(a,b) = (a+b, -a-b)$

$\operatorname{Ker} f_1 = \{(a,-a)\} \cong \mathbb{Z}$

$$\operatorname{im}(g_1) \cong H_1(A)\oplus H_1(B)/_{\ker g_1}$$

$$\cong \mathbb{Z}\oplus\mathbb{Z}/_{\{(c,-c)\}}$$

$$\cong \mathbb{Z}$$

$$[(a,b)] = [(a+b,0)]$$

$$\overset{\shortparallel}{[(a,b)+(b,-b)]}$$

Similarly

$f_0(a,b) = (a+b, -a-b)$

$\operatorname{Ker} f_0 = \{(a,-a)\} \cong \mathbb{Z}$

$\operatorname{Im} f_0 = \{(c,-c)\}$

Exactness: $\operatorname{im}\partial_0 = \ker f_0$

$\operatorname{im} g_1 = \ker \partial_0$

$$H_1(X)/_{\ker\partial_0} \cong \operatorname{im}\partial_0 = \ker f_0 \cong \mathbb{Z}$$

$\operatorname{im} g_1 \cong \mathbb{Z}$ $\qquad H_1(X)/_{\mathbb{Z}} \cong \mathbb{Z}$ $\qquad \rightsquigarrow \quad H_1(X) \cong \mathbb{Z}\oplus\mathbb{Z}$

Connected sum:

Given two K-diml manifolds M^k, N^k (closed, oriented) connected

(near every pt there is a nbhd homeo to an open subset of $\mathbb{R}^k$)

Remember: $H_K(M^k) \cong \mathbb{Z}$

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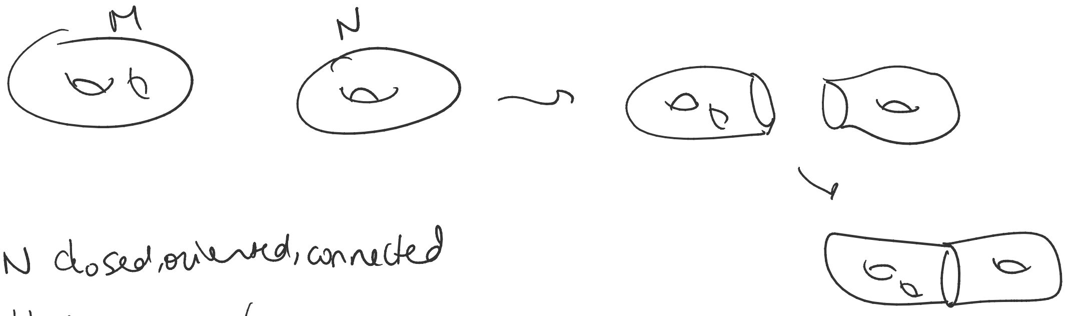
Connected sum: $M \# N = (M - D_M) \underset{\partial D_M = \partial D_N}{\cup} (N - D_N)$

delete a small ball from M

delete a small ball from N

resulting spaces each have boundary $\cong S^{k-1}$ & we identify these two S^{k-1}'s.



M, N closed, oriented, connected

Theorem: $$H_n(M \# N) \cong \begin{cases} H_n(M) \oplus H_n(N) & n \neq K \\ \mathbb{Z} & n = K \end{cases}$$

We will prove this by:

$H_n(M - D_M)$, $H_n(N - D_N)$ > using exact seq of pair, excision

$H_n(M \# N)$ — by Mayer-Victoris.