Lecture 2

Wednesday, January 6, 2021 1:54 PM

Advantage: finite data to use to construct homology

Disadvantage: a priori depends on choice of Δ-complex structure on X

Next time: singular homology will have opposite adv/dis

Simplicial homology

Homology algebraic construction takes as input

a chain complex $\leftarrow$ 2 parts:

① a graded module C

$$C = \bigoplus_{n \in \mathbb{Z}} C_n$$

② differential map $d : C \to C$

decreases grading by 1

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[rr, bend right] & C_{n-1} \arrow[r, "d_{n-1}"] & C_{n-2} \arrow[r] & \cdots \end{tikzcd}$$

one rule: $d \circ d = 0$ $\quad d_{n-1} \circ d_n = 0$ (for each n)

From this can define homology:

n^{th} homology group: $H_n = \ker d_n / \operatorname{im} d_{n+1}$

$\ker d_n \subseteq C_n$

$\operatorname{im} d_{n+1} \subseteq C_n$

Need $\operatorname{im}(d_{n+1}) \subseteq \ker(d_n)$

Something in $\operatorname{im}(d_{n+1})$: $\quad d_{n+1}(x)$

Is it in $\ker(d_n)$? $\quad d_n(d_{n+1}(x)) \stackrel{?}{=} 0$

Rule $\quad d_n \circ d_{n+1} = 0$ ensures $\operatorname{im}(d_{n+1}) \subseteq \ker(d_n)$

so H_n is well defined.

Simplicial chain complex: Fixed X with Δ-complex structure

Modules: $\quad C_n^{\Delta}(X)$ free module over $\mathbb{Z}$ generated by

$\{\sigma_\alpha : \Delta^n \to X\}$ $\leftarrow$ from Δ-complex structure

In particular, $C_n^{\Delta}(X) = 0$ when $n < 0$.

Addition is formal: $\quad \sigma_1 + \sigma_2$ doesn't represent another σ_α

In particular, $C_n(X) = 0$

Addition is formal : $\sigma_1 + \sigma_2$ doesn't represent another σ_α

(roughly imagine $\sigma_1 + \sigma_2$ as union of images in X)

$3\sigma_1$

$$C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} = 0$$

$$\ker d_0 / \operatorname{im} d_1$$

<u>Differential</u>: "Boundary map"

$d_n : C_n \longrightarrow C_{n-1}$ define d_n on generators + extend linearly

$\sigma_\alpha : \Delta^n \longrightarrow X$

$$d_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \Big|_{\underbrace{[v_0 \cdots \hat{v}_i \cdots v_n]}_{i^{th} \text{ face of } \Delta^n}}$$

a generator of C_{n-1} by defn of Δ-complex

$v_i \in \Delta^n$
where ith coord is 1
all other coords are 0

convex hull spanned by all vertices except v_i

For this to satisfy the rule need to check

$\underline{d_{n-1} \circ d_n = 0}$ the alternating signs are important here

<u>Lemma</u>: $d_{n-1} \circ d_n = 0$

<u>Proof</u>: Check on generators : $\sigma_\alpha : \Delta^n \longrightarrow X$

$$d_{n-1}(d_n(\sigma_\alpha)) = d_{n-1}\left(\sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \Big|_{[v_0 \cdots \hat{v}_i \cdots v_n]} \right)$$

$$= \sum_{i=0}^{n} (-1)^i \, d_{n-1}\left(\underbrace{\sigma_\alpha \Big|_{[v_0 \cdots \hat{v}_i \cdots v_n]}}_{\Delta^{n-1} \longrightarrow X} \right)$$

$$= \sum_{i=0}^{n} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j \, \sigma_\alpha \Big|_{[v_0 \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n]} \right.$$

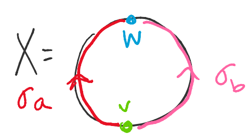
$$\left. + \sum_{j=i+1}^{n} (-1)^{j-1} \, \sigma_\alpha \Big|_{[v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]} \right)$$

$$+\sum_{j=i+1} (\;)\;\sigma_\alpha\big|_{[v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]} \;)$$

$$= \underbrace{\sum_{i=0}^{n}\sum_{j=0}^{i-1} (-1)^{i+j}\,\sigma_\alpha\big|_{[v_0 \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n]}} + \underbrace{\sum_{i=0}^{n}\sum_{j=i+1}^{n} (-1)^{i+j-1}\,\sigma_\alpha\big|_{[v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]}}$$

These differ by a factor of (-1)

$$= 0$$

Example: $X =$ 

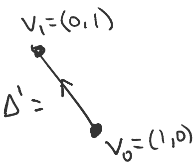
Δ-complex

$\sigma_v : \Delta^0 \to X \qquad \sigma_v(v_0) = v$

$\sigma_w : \Delta^0 \to X \qquad \sigma_w(v_0) = w$

$C_1^\Delta(X) = \mathbb{Z}\langle \sigma_a \rangle \oplus \mathbb{Z}\langle \sigma_b \rangle$

$C_0^\Delta(X) = \mathbb{Z}\langle \sigma_v \rangle \oplus \mathbb{Z}\langle \sigma_w \rangle$

$\Delta^1 =$ 

$\Delta^0 = \{1\} \subset \mathbb{R}$

$$C_1^\Delta(X) \xrightarrow{d_1} C_0^\Delta(X) \xrightarrow{d_0} 0$$

$$d_1(\sigma_a) = \sigma_a\big|_{[(0,1)]} - \sigma_a\big|_{[(1,0)]}$$
$$= \sigma_w - \sigma_v$$

Similarly $\quad d_1(\sigma_b) = \sigma_w - \sigma_v$

$$d_0(\sigma_v) = 0 \qquad d_0(\sigma_w) = 0$$

We will finish the calculation of homology in this example on Friday, but go ahead and try it yourself before then.