Lecture 20

Wednesday, February 24, 2021 2:04 PM

Calculate homology under cut + paste operations:

① Connected sum 2 connected k-dim'l manifolds M, N

$$M \# N = (M \setminus \mathring{D}_M) \underset{\partial D_M = \partial D_N}{\cup} (N \setminus \mathring{D}_N)$$

Strategy to calculate $H_n(M\#N)$:

3. Mayer-Vietoris relates $H_*(\underbrace{M\#N}_{X})$ to $H_*(\overbrace{M\setminus\mathring{D}_M}^{A})$, $H_*(\overbrace{N\setminus\mathring{D}_N}^{B})$ and $H_*(\underbrace{\partial D_M}_{A\cap B})$

2. Calculate $H_*(M\setminus\mathring{D}_M)$ in terms of $H_*(M)$ using exact sequence of pair $(M\setminus\mathring{D}_M, \partial D_M)$

   missing term is $H_*(M\setminus\mathring{D}_M, \partial D_M)$

   $\cong$ excision

   $H_*(M, D_M)$

1. Calculate $H_*(M, D_M)$ in terms of $H_*(M)$ using exact seq of pair (M, D_M)

Step 1:

$n>1$

$$\cdots \to \underset{0}{\underset{\|}{H_n(D_M)}} \to H_n(M) \xrightarrow{\cong} H_n(M, D_M) \to \underset{0}{\underset{\|}{H_{n-1}(D_M)}} \to \cdots$$

exactness

$H_n(M) \cong H_n(M, D_M)$ for $n>1$

iso when both pieces of pair are connected

$$\to 0 \to H_1(M) \xrightarrow{\cong} H_1(M, D_M) \xrightarrow{0} \underset{\mathbb{Z}}{H_0(D_M)} \underset{\cong}{\xrightarrow{i_*}} \underset{\mathbb{Z}}{H_0(M)} \to H_0(M, D_M) \to 0$$

$H_1(M) \cong H_1(M, D_M)$

$H_0(M, D_M) = 0$.

Step 2: Excision: $H_n(M\setminus\mathring{D}_M, \partial D_M) \cong H_n(M, D_M) \cong \tilde{H}_n(M)$

Step 2: Excision: $\underline{H_n(M\setminus \mathring{D}_M, \partial D_M)} \cong \boxed{H_n(M, D_M) \cong \tilde{H}_n(M)}$

$\partial D_M \cong S^{k-1}$

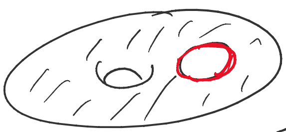
Exact seq of $(M\setminus\mathring{D}_M, \partial D_M)$ $\qquad H_n(\partial D_M)\cong \begin{cases}\mathbb{Z} & n=k-1,0\\ 0 & \text{else}\end{cases}$

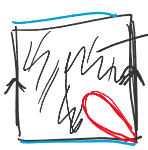
$$\cdots\to \underset{\overset{\|}{0}}{H_n(\partial D_M)}\to H_n(M\setminus\mathring D_M)\xrightarrow{\cong} H_n(M\setminus \mathring D_M,\partial D_M)\to \underset{\overset{\|}{0}}{H_{n-1}(\partial D_M)}\to\cdots$$

$n, n-1 \neq 0, k-1$ $\qquad H_n(M\setminus\mathring D_M)\cong H_n(M\setminus\mathring D_M,\partial D_M)\cong\tilde H_n(M)$

$$\underset{\overset{\|}{0}}{H_k(\partial D_M)}\to \underset{\overset{\|}{0}}{H_k(M\setminus\mathring D_M)}\xhookrightarrow{0} \underset{\overset{\text{\rotatebox{90}{$\cong$}}}{H_k(M)\cong\mathbb{Z}}}{H_k(M\setminus\mathring D_M,\partial D_M)}\underset{\cong}{\xrightarrow{\partial}} \underset{\overset{\|}{\mathbb{Z}}}{H_{k-1}(\partial D_M)}\xrightarrow{0} H_{k-1}(M\setminus \mathring D_M)\xrightarrow{\cong} \underset{\overset{\text{\rotatebox{90}{$\cong$}}}{H_{k-1}(M)}}{H_{k-1}(M\setminus\mathring D_M,\partial D_M)}\xrightarrow{0}$$

$$\partial: \underset{\overset{\text{\rotatebox{90}{$\cong$}}}{\mathbb{Z}}}{H_k(M\setminus\mathring D_M,\partial D_M)}\to \underset{\overset{\|}{\mathbb{Z}}}{H_{k-1}(\partial D_M)}$$





e_2

$[e_2]\in H_2(T^2\setminus D, \partial D)$

$d_2(e_2) = O = \partial(e_2)\in H_1(\partial D)$, $= 0$

is modded out in $C_1(M\setminus D_M,\partial D_M) = C_1(M\setminus D_M)/C_1(\partial D_M)$

Representative of generator is a triangulation of $M\setminus\mathring D_M$

generator of is triangulation of ∂D_M

(this is a relative cycle b/c its boundary is ∂D_M)

$H_k(M\setminus\mathring D_k)=0$ $\qquad H_n(M\setminus\mathring D_M)\cong H_n(M)$ for $n\neq k$

③ **Gluing:** Mayer Vietoris exact sequence $\qquad A = M\setminus\mathring D_M$, $B=N\setminus\mathring D_N$ $\qquad A\cap B\simeq S^{k-1}$

$$\cdots\to \underset{\overset{\|}{0}}{H_n(S^{k-1})}\to H_n(M\setminus\mathring D_M)\oplus H_n(N\setminus\mathring D_N)\to H_n(M\# N)\to \underset{\overset{\|}{0}}{H_{n-1}(S^{k-1})}\to\cdots$$

If $n,n-1\neq 0,k-1$ exactness $\Rightarrow$ $H_n(M\# N)\cong H_n(M\setminus\mathring D_M)\oplus H_n(N\setminus\mathring D_N)$

$\cong H_n(M)\oplus H_n(N)$

$n, n-1 = 0$:

$$0 \to H_1(M\setminus \mathring{D}_M) \oplus H_1(N \setminus \mathring{D}_N) \xrightarrow{\cong} H_1(M\#N) \xrightarrow{0} H_0(S^{k-1}) \xhookrightarrow{f_0} H_0(M\setminus D_M) \oplus H_0(N\setminus D_N) \xrightarrow{g_0} H_0(M\#N) \to 0$$

$H_0(S^{k-1}) \cong \mathbb{Z}$, $H_0(M\setminus D_M) \cong \mathbb{Z}$, $H_0(N\setminus D_N) \cong \mathbb{Z}$, $H_0(M\#N) \cong \mathbb{Z}$

$g_0(x, y) = x + y$

$f_0(a) = (a, -a)$

$H_n(M\#N) = H_n(M) \oplus H_n(N)$ for $n = 1, \dots, k-2$

$H_0(M\#N) = \mathbb{Z}$

$n, n-1 = k-1$

$$0 \to H_k(M\setminus D_M) \oplus H_k(N\setminus D_N) \xhookrightarrow{g_k} H_k(M\#N) \xrightarrow[\cong]{\partial} H_{k-1}(S^{k-1}) \xrightarrow{f_{k-1} = 0} H_{k-1}(M\setminus D_M) \oplus H_{k-1}(N\setminus D_N) \xrightarrow[\cong]{g_{k-1}} H_{k-1}(M\#N) \to 0$$

$H_k(M\setminus D_M) = 0$, $H_k(N\setminus D_N) = 0$, $H_k(M\#N) \leftarrow \mathbb{Z}$, $H_{k-1}(S^{k-1}) \cong \mathbb{Z}$

$$f_{k-1} : H_{k-1}(S^{k-1}) \to H_{k-1}(M\setminus D_M) \oplus H_{k-1}(N\setminus D_N)$$

$\mathbb{1} \in H_{k-1}(S^{k-1})$

$$f_{k-1}(\mathbb{1}) = ([\partial D_M], -[\partial D_N]) = (0, 0)$$

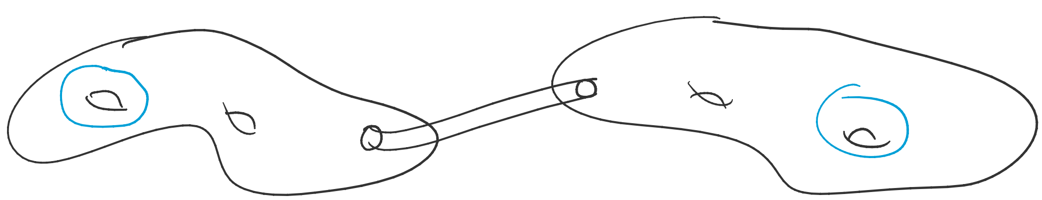
Is 0 as a k-1 homology class in $H_{k-1}(M\setminus D_M)$

$\partial D_M \in \operatorname{Im} d_{n+1}^{M\setminus D_M}$

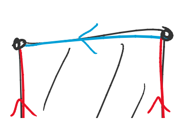
$\partial D_M = d_{n+1}^{M\setminus D_M}(M\setminus \mathring{D}_M)$

$H_k(M\#N) \cong \mathbb{Z}$ $\quad$ $H_n(M\#N) \cong \underline{H_n(M) \oplus H_n(N)}$ $\quad$ $n \neq 0, k$

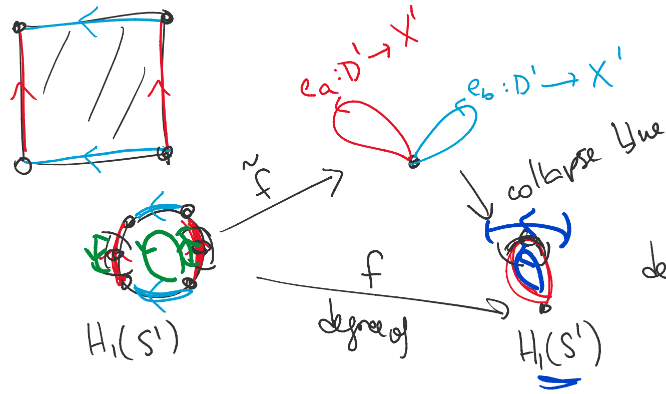
$H_0(M\#N) \cong \mathbb{Z}$



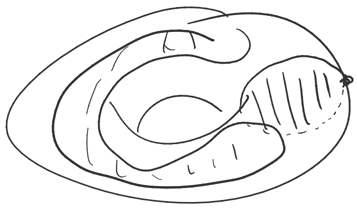
Dehn surgery on a knot next time.



$e_a : D' \to X'$ $\quad$ $e_b : D' \to X'$



$\deg f = \pm 1 \quad \pm 1$



HW5 2b

$$\mathbb{R}^n \cong \underbrace{[1 : x_1 : \dots : x_n]} \longrightarrow \underbrace{[0 : x_1 : \dots : x_n]}$$

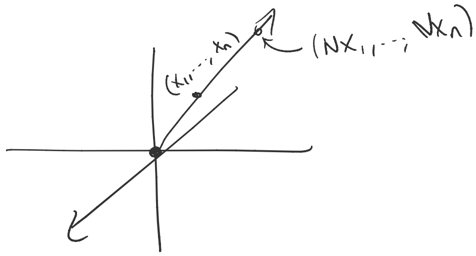
$(x_1, \dots, x_n)$

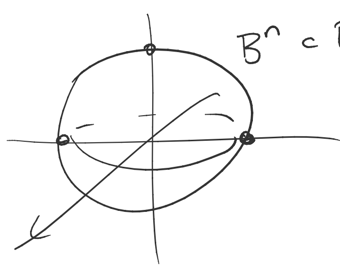
$$\left[\tfrac{1}{N} : x_1, \dots, x_n\right] \xrightarrow[N \to \infty]{} \underbrace{[0 : x_1 : \dots : x_n]}$$

$$\| $$

$$[1 : Nx_1, \dots, Nx_n]$$

$\mathbb{R}^n$





$B^n \subset \mathbb{R}^n \quad (x_1, \dots, x_n)$

glue points in $S^{n-1} = \partial D^n$

to $[0 : x_1 : \dots : x_n] \in \mathbb{R}P^{n-1}$

degree of gluing map: understand

$$f : S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$$

$$(x_1, \dots, x_n) \longmapsto [0 : x_1 : \dots : x_n]$$

$$f^{-1}([0 : 1 : 0 : \dots : 0])$$