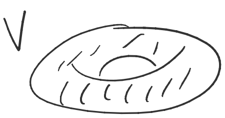
Lecture 21

Friday, February 26, 2021 1:59 PM



Dehn surgery:

Input: • Knot $K \subset S^3$ ← embedding of S^1

• rational number p/q

Construction: Let N be a neighborhood of K

← $M_K := S^3 \setminus N$ ← Knot complement

$V = S^1 \times D^2$

p/q Dehn surgery on K → $S^3_{p/q}(K) = M_K \cup V$

$\cup$ ← glued (along boundary) according to p/q as follows:

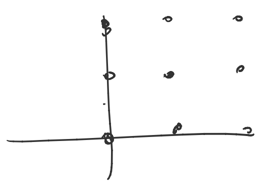
boundary

↓

$\partial M_K \cong T^2$ $\qquad f: T^2 \to T^2$ homeomorphism related to p/q

$\partial V \cong T^2$

$T^2 = \mathbb{R}^2/\mathbb{Z}^2$



2x2 matrix invertible takes $\mathbb{Z}^2$ to $\mathbb{Z}^2$ & inverse takes $\mathbb{Z}^2$ to $\mathbb{Z}^2$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad a,b,c,d \in \mathbb{Z}$ ← ensures $\mathbb{Z}^2 \to \mathbb{Z}^2$

$\big\downarrow{}^{-1}$

$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ ← has $\mathbb{Z}$ entries if $ad-bc = \pm 1$

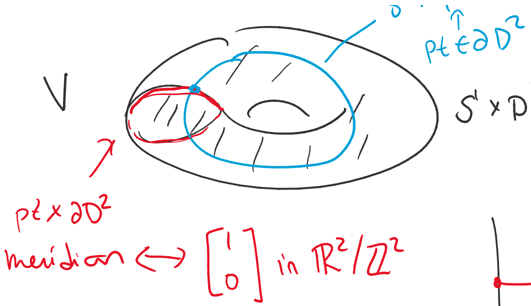
a good class of homeomorphisms $T^2 \to T^2$ are

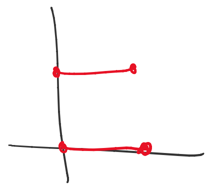
$SL(2,\mathbb{Z})$ ← 2x2 integer matrices with det = 1.

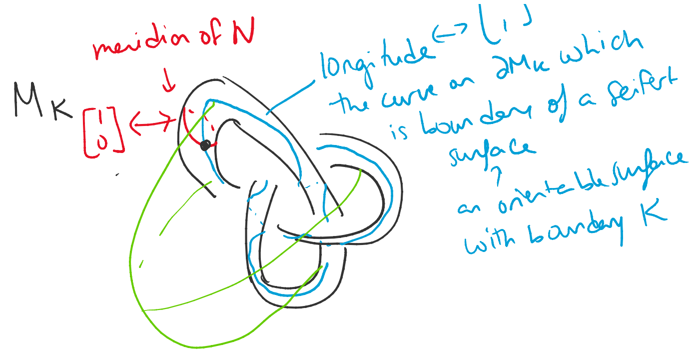
$f = \begin{bmatrix} p & p' \\ q & q' \end{bmatrix}$ s.t. $pq' - p'q = 1$ (if p, q rel prime then such p', q' exist).

Important to clarify how I identify ∂M_K with $\mathbb{R}^2/\mathbb{Z}^2$

and ∂V with $\mathbb{R}^2/\mathbb{Z}^2$





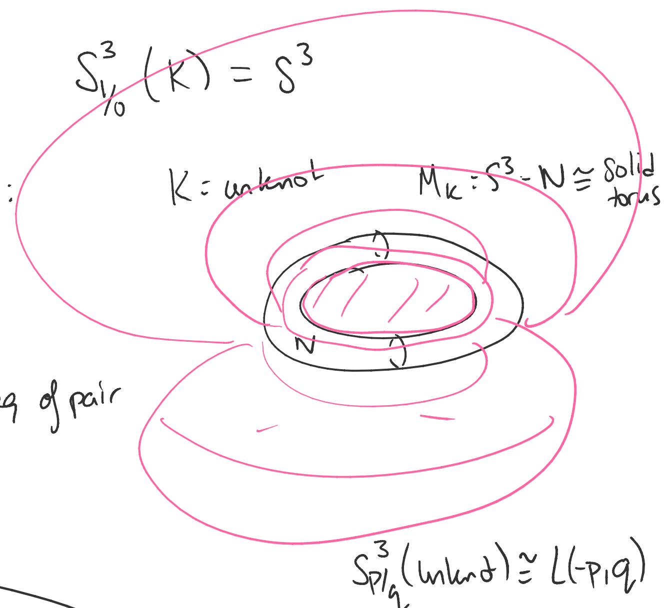


Identify a point $\begin{bmatrix}x\\y\end{bmatrix} \in \partial V$ with $\begin{bmatrix}p & p'\\q & q'\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ in ∂M_K

$= \begin{bmatrix}px+p'y\\qx+q'y\end{bmatrix}$

$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ glues V into M_K just as N was $\quad S^3_{1/0}(K) = S^3$

$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ glues V into M_K in a different way:



Goal: Calculate $H_n(S^3_{p/q}(K))$

Strategy from Wednesday: cut - exact seq of pair

Every or cl 3-mfld is Dehn surgery on a link.

cut out $S^k \times D^{n-k} \xrightarrow{\partial} S^k \times S^{n-k-1}$

glue in $D^{k+1} \times S^{n-k-1}$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{surgery}}$

in dim 3 $n=$ $\quad S^1 \times D^2$, $\quad D^2 \times S^1$

Find $H_n(M_K)$ $\quad S^3 \setminus N$:

$$H_3(N) \to H_3(S^3) \xrightarrow{\cong} H_3(S^3,N) \to H_2(N) \to H_2(S^3) \to H_2(S^3,N) \xrightarrow{\cong} \underset{\cong\,\mathbb{Z}}{H_1(N)} \to \underset{=\,0}{H_1(S^3)}$$

$$\underset{=0}{H_3(N)} \to \underset{\cong\mathbb{Z}}{H_3(S^3)} \xrightarrow{\cong} H_3(S^3,N) \to \underset{=0}{H_2(N)} \to \underset{=0}{H_2(S^3)} \to H_2(S^3,N) \xrightarrow{\cong} \underset{\cong\mathbb{Z}}{H_1(N)} \to \underset{=0}{H_1(S^3)}$$

$$\xrightarrow{0} H_1(S^3,N) \xrightarrow{0} \underset{\cong\mathbb{Z}}{H_0(N)} \xrightarrow{\cong} \underset{\cong\mathbb{Z}}{H_0(S^3)} \xrightarrow{0} H_0(S^3,N) \to 0$$

$H_3(S^3,N) \cong \mathbb{Z}$ $\quad$ $H_1(S^3,N) = 0$ $\qquad$ $S^3 - \mathring{N} = M_K$

$H_2(S^3,N) \cong \mathbb{Z}$ $\quad$ $H_0(S^3,N) = 0$ $\qquad$ $N - \mathring{N} = \partial M_K$

Excision: $H_3(M_K, \partial M_K) \cong \mathbb{Z}$

$H_2(M_K, \partial M_K) \cong \mathbb{Z}$ $\qquad$ $\partial M_K \cong T^2$

$H_1(M_K, \partial M_K) = 0$

H_0 $\quad$ "

Want $H_n(M_K)$:

$$\underset{=0}{H_3(T^2)} \to \underset{=0}{H_3(M_K)} \xrightarrow{0} \underset{\cong\mathbb{Z}}{H_3(M_K,\partial M_K)} \xrightarrow[\cong]{\partial^3} \underset{\cong\mathbb{Z}}{H_2(T^2)} \xrightarrow{0} \underset{=0}{H_2(M_K)} \xrightarrow{0} \underset{\cong\mathbb{Z}}{H_2(M_K,\partial)} \xrightarrow{\partial^2} \underset{\cong\mathbb{Z}^2}{H_1(T^2)} \twoheadrightarrow \underset{\cong\mathbb{Z}}{H_1(M_K)} \to \underset{=0}{H_1(M_K,\partial)}$$

$$\to \underset{\cong\mathbb{Z}}{H_0(T^2)} \to \underset{\cong\mathbb{Z}}{H_0(M_K)} \to \underset{=0}{H_0(M_K,\partial)}$$

$a \longmapsto (0,a)$

$H_1(M_K) \cong H_1(T^2) / \{(0,a)\} \cong \mathbb{Z}^2/_{0\times\mathbb{Z}} \cong \mathbb{Z}$

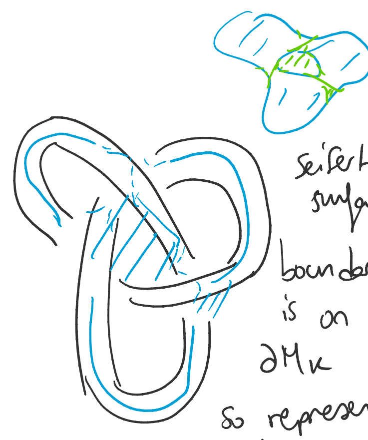
Claim: ∂^3 & ∂^2 are injective, ∂^3 surjective

$\partial^3 : H_3(M_K, \partial M_K) \to H_2(\partial M_K)$

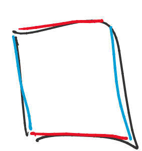
generated by $M_K \mapsto \partial M_K \leftarrow$ generates $H_2(\partial M_K) = H_2(T^2)$

$\partial^2 : \underset{\cong\mathbb{Z}}{H_2(M_K, \partial M_K)} \to H_1(\partial M_K)$

[Seifert surface] $\longmapsto$ longitude $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



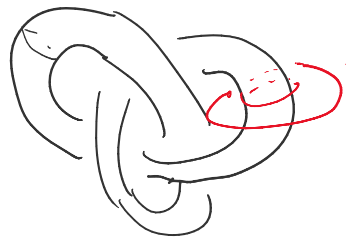
Seifert surface boundary is on ∂M_K so represents a relative cycle



$H_3(M_K) \cong H_2(M_K) \cong 0$

$H_1(M_K) \cong \mathbb{Z}$ $\quad$ $H_0(M_K) \cong \mathbb{Z}$

Gluing will record something about p/q $\quad$ Mayer Vietoris $\rightsquigarrow$ $H_n(S^3_{p/q}(K))$



$S^3 \setminus K$ ← non-compact

$S^3 \setminus \mathring{N}$ ← compact version
homotopy equivalent

A $S^2 \hookrightarrow X^4 \leftarrow$ 4D mfld

nbhd is locally $S^2 \times D^2$

but globally a D^2 bundle over S^2