Lecture 22

Monday, March 1, 2021 2:07 PM

<u>Last time</u>: For any knot $K \subset S^3$ with regular neighborhood $N \subset S^3$

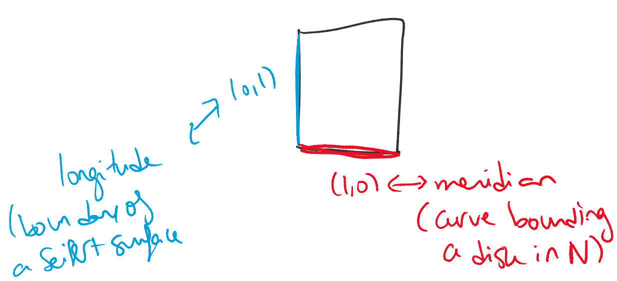
$$H_n(S^3 - N) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

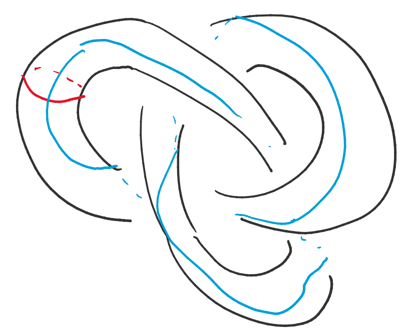
$M_K = S^3 - N$ $\qquad\qquad$ $V = D^2 \times S^1 \leftarrow$ solid torus

<u>Dehn surgery</u>: $S^3_{p/q}(K) = M_K \cup V$

glued along boundaries via homeomorphism $\begin{bmatrix} p & p' \\ q & q' \end{bmatrix}$ where $pq' - qp' = 1$

Recall we identified the boundary of M_K with T^2 by:
$\qquad\qquad$ $\shortparallel$
$\qquad$ boundary of N





We identify boundary of V with T^2

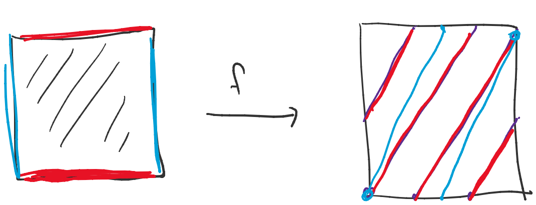
$V = D^2 \times S^1 \qquad\qquad \partial V = \partial D^2 \times S^1$

$$\partial D^2 \times pt \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$pt \times S^1 \longleftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f : \underset{(T^2)}{\partial V} \xrightarrow{\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}} \underset{(T^2)}{\partial M_K}$$

how does this homeomorphism act on homology?

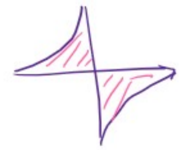


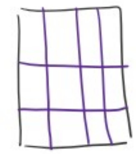
$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$
$$= p\begin{pmatrix} 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$H_0(T^2) \cong \mathbb{Z}$

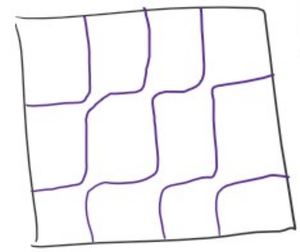
$H_1(T^2) \cong \mathbb{Z}^2 \leftarrow$ generators are $\begin{pmatrix}1\\0\end{pmatrix}$ + $\begin{pmatrix}0\\1\end{pmatrix}$

$H_2(T^2) \cong \mathbb{Z} \leftarrow$ generated by the 2-cell





$p=2 \quad q=3$



$\begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}$

$\begin{pmatrix}p\\q\end{pmatrix}$ as a $H_1(T^2)$ class is represented by a curve that goes around T^2 p times horiz + q times vertically

$\hookrightarrow$ a curve of slope q/p in $\mathbb{R}^2/\mathbb{Z}^2$

$f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}p'\\q'\end{pmatrix}$

$f_*: H_1(T^2) \to H_1(T^2)$ is $f_* = \begin{pmatrix}p & p'\\q & q'\end{pmatrix}$

$\left.\begin{array}{l} f_*: H_0(T^2) \to H_0(T^2) \\ f_*: H_2(T^2) \to H_2(T^2) \end{array}\right\}$ identity

<u>Glue:</u> Mayer-Vietoris $\quad A = M_K \quad B = V \quad X = S^3_{p/q}(K)$

$A \cap B = T^2 \leftarrow$ identified boundaries

$V = D^2 \times S^1 \simeq S^1$ ← htpy

$$\underset{0}{H_3(T^2)} \to \underset{0}{H_3(M_K)} \oplus \underset{0}{H_3(V)} \to \underset{\cong\,\mathbb{Z}}{H_3(S^3_{p/q}(K))} \xrightarrow{\cong} \underset{\cong\,\mathbb{Z}}{H_2(T^2)} \to \underset{0}{H_2(M_K)} \oplus \underset{0}{H_2(V)}$$

$$0 \hookrightarrow H_2(S^3_{p/q}(K)) \hookrightarrow \underset{\cong\,\mathbb{Z}^2}{H_1(T^2)} \xrightarrow{\alpha} \underset{\cong\,\mathbb{Z}}{H_1(M_K)} \oplus \underset{\cong\,\mathbb{Z}}{H_1(V)} \to H_1(S^3_{p/q}(K))$$

$$0 \hookrightarrow \underset{\cong\,\mathbb{Z}}{H_0(T^2)} \hookrightarrow \underset{\cong\,\mathbb{Z}}{H_0(M_K)} \oplus \underset{\cong\,\mathbb{Z}}{H_1(V)} \to \underset{\cong\,\mathbb{Z}}{H_0(S^3_{p/q}(K))} \to 0$$

$a \longmapsto (a, -a)$

$(x, y) \longmapsto x+y$

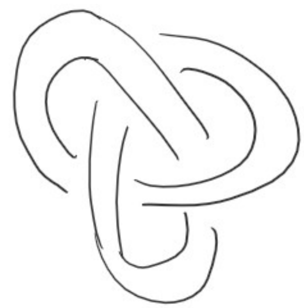
$$0 \longrightarrow H_2(S^3_{p/q}(K)) \hookrightarrow \underset{\cong\,\mathbb{Z}^2}{H_1(T^2)} \xrightarrow{\alpha} \underset{\cong\,\mathbb{Z}}{H_1(M_K)} \oplus \underset{\cong\,\mathbb{Z}}{H_1(V)} \xrightarrow{\beta} H_1(S^3_{p/q}(K)) \xrightarrow{0}$$

$$0 \longrightarrow \underbrace{H_2(S^3_{p/q}(K))}\hookrightarrow \underset{\cong \mathbb{Z}^2}{H_1(T^2)} \xrightarrow{\alpha} \underset{\cong \mathbb{Z}}{H_1(M_K)} \oplus \underset{\cong \mathbb{Z}}{H_1(V)} \xrightarrow{\beta} H_1(S^3_{p/q}(K)) \longrightarrow$$

$$H_2(S^3_{p/q}(K)) \cong \operatorname{Ker}\alpha = \begin{cases} 0 & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}$$

$$\underline{H_1(S^3_{p/q}(K))} = \operatorname{Im}\beta \cong H_1(M_K)\oplus H_1(V) / \operatorname{Ker}\beta = \underline{H_1(M_K)\oplus H_1(V) / \operatorname{im}\alpha}$$

<u>What is α?</u> $\alpha(a,b) = (i_{M_K *}(a,b), -i_{V_*}(a,b))$



$\binom{a}{b}$ = a knot's meridian + b knot's long

in $H_1(M_K)$ longitude $(0,1)$ is 0

meridian $(1,0)$ generates

$i_{M_K *}(a,b) = a \in H_1(M_K)$

Understand $(a,b) \in T^2$ from perspective of V

$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} : \partial V \longrightarrow \partial M_K$

Inverse homeomorphism: $\begin{pmatrix} q' & -p' \\ -q & p \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} q'a - p'b \\ -qa + pb \end{pmatrix}$ — # $\partial D^2 \times pt$ / # $pt \times S^1$ in V

In $H_1(V)$: generated by $pt \times S^1$

and $\partial D^2 \times pt$ is 0 in $H_1(V)$

$i_{V *}(a,b) = -qa + pb$

$$\alpha(a,b) = (a, -qa + pb)$$

$\operatorname{Ker}(\alpha)$: $\alpha(a,b) = 0 \iff a = 0$ + $-qa + pb = 0$

$\iff a = 0$, $pb = 0$

If $p \neq 0$ $\operatorname{Ker}\alpha = 0$

If $p = 0$ $\operatorname{Ker}\alpha = \{(0,b)\} \cong \mathbb{Z}$

$\mathrm{im}(\alpha) = \{(a, -qa+pb) \mid a,b \in \mathbb{Z}\}$

$$H_1(X) \cong H_1(M_K) \oplus H_1(V) / \mathrm{im}\,\alpha = \mathbb{Z}\oplus\mathbb{Z} \Big/ \{(a,-qa+pb) \mid a,b \in \mathbb{Z}\}$$

$$= \mathbb{Z}\langle (1,0),(0,1)\rangle \Big/ \mathbb{Z}\langle (1,-q),(0,p)\rangle$$

$$= \mathbb{Z}\langle (\underline{1},-q),(0,1)\rangle \Big/ \mathbb{Z}\langle (\underline{1,-q}), \underline{p}(0,1)\rangle$$

$$\cong \mathbb{Z}/p\mathbb{Z}$$

If $p=0$ $\mathbb{Z}$

<u>Conclusion:</u>

$$H_n(S^3_{p/q}(K)) = \begin{cases} \mathbb{Z} & n=0,3 \\ 0 & n=2 \\ \mathbb{Z}/p\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

$p \neq 0$ case

If $p=0$
$$H_n(\underline{S^3_{p/q}}(K)) = \begin{cases} \mathbb{Z} & n=0,1,2,3 \\ 0 & \text{else} \end{cases}$$

<u>Observation:</u> This calculation does not depend on K, q, p', q'

<u>Topology fact:</u> Homeomorphism type generally does depend on K, q

π_1 might depend on K

Homology with other coefficients

For any ring or abelian group G we can define

$C_n(X;G) = G\langle n\text{-simplices}\rangle$ i.e. $\sum n_i \sigma_i$

d_n^G defined some way

in G

$\rightsquigarrow \quad H_n(X;G) = \ker d_n^G / \operatorname{im} d_{n+1}^G$

Have relative version, excision, exact seq of pair, cellular homology $C_n^{cw}(X;G) = G\langle n\text{-cells}\rangle$

in G-versions.

Depending on G you choose, $H_n(X;G)$ may pick up different info about X

Common choices for G: $\mathbb{Z}_p$ esp. $\mathbb{Z}_2$

$\mathbb{Q}$

$\mathbb{R}$

$\mathbb{Z}[\pi_1(X)]$