Lecture 23

Wednesday, March 3, 2021   2:06 PM

Homology with coefficients: chain complex freely generated by singular chains over any abelian group G (instead of $\mathbb{Z}$).

$H_n(X; G)$

Example: $\mathbb{RP}^n$ cellular chain complex

$$C_k(\mathbb{RP}^n; G) \cong \begin{cases} G & 0 \le k \le n \\ 0 & \text{else} \end{cases} \quad \text{(one cell of dim } 0, 1, \dots, n)$$

$$\to 0 \to \underbrace{G \xrightarrow{\times 2} G \xrightarrow{0} G}_{\text{maybe shifted if } n \text{ even vs odd}} \xrightarrow{\times 2} \cdots \xrightarrow{0} G \xrightarrow{\times 2} G \xrightarrow{0} G \to 0 \to$$

On hw you found d_n by calculating degrees

If $G = \mathbb{Z}/2\mathbb{Z}$ then $\times 2 = 0$

So $$H_k(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } 0 \le k \le n \\ 0 & \text{else} \end{cases}$$

If $G = \mathbb{Q}$ then $\times 2$ is an isomorphism (im $= \mathbb{Q}$, ker $= 0$)

$$0 \to \mathbb{Q} \xrightarrow{\times 2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\times 2} \cdots \to \mathbb{Q} \xrightarrow{\times 2} \mathbb{Q} \xrightarrow{0} \underset{C_0}{\mathbb{Q}} \to 0 \to \cdots$$

n even

$$C_{\text{even}}(\mathbb{RP}^n) \xrightarrow{\times 2} C_{\text{odd}}(\mathbb{RP}^n)$$

If n even $$H_k(\mathbb{RP}^n; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k = 0 \\ 0 & \text{else} \end{cases}$$

n odd

$$0 \to \underset{C_n}{\mathbb{Q}} \xrightarrow{0} \mathbb{Q} \xrightarrow{\times 2} \cdots \qquad H_k(\mathbb{RP}^n; \mathbb{Q}) : \begin{cases} \mathbb{Q} & k = 0, n \\ 0 & \text{else} \end{cases}$$

$G = \mathbb{Z}$ n even

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k \in \{1, \dots, n\} \text{ \& odd} \end{cases}$$

$G = \mathbb{Z}$ $\quad$ n even

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2 & k \in \{1,\dots,n\} \text{ \& odd} \\ 0 & \text{else} \end{cases}$$

n odd

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \qquad H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0,n \\ \mathbb{Z}/2 & k \in \{1,\dots,n-1\} \text{ \& odd} \\ 0 & \text{else} \end{cases}$$

Axioms for homology theory

(algebraic topology of topological spaces) NOT homological algebra notion of homology theory (coming from any chain complex)

For all CW pairs (X,A) (absolute case $(X,\emptyset)$, reduced (X, pt))

"homology theory"

CW pair $\to (X,A) \longrightarrow \{h_n(X,A)\}_{n\in\mathbb{Z}}$ ← sequence of abelian groups

$f:(X,A)\to(Y,B) \longrightarrow f_*: h_n(X,A) \to h_n(Y,B)$ for each $n \in \mathbb{Z}$

satisfying $(f\circ g)_* = f_* \circ g_*$

$(id)_* = id$

Satisfying the following axioms:

1. homotopy invariance: if $f \simeq g$ then $f_* = g_*$.

2. $\exists$ long exact sequences of pairs $\quad (h_n(X) := h_n(X,\emptyset))$

$$\dots \to h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X,A) \xrightarrow{\partial} h_{n-1}(A) \longrightarrow \dots$$

3. Excision for any (X,A) and U s.t. $\bar{U} \subset \mathring{A}$

$i_*: h_n(X-U, A-U) \to h_n(X,A)$ is an isomorphism.

4. Additivity $\quad h_n(\bigsqcup_\alpha X_\alpha) = \bigoplus_\alpha h_n(X_\alpha)$

5. $h_n(pt) = 0$ for all $n \neq 0$.

Theorem: Any such homology theory is isomorphic to $H_n(X;G)$ where $G = h_0(pt)$.

Lemma: Any homology theory has

$$\underline{h_n(S^k)} \cong \begin{cases} G & n = 0, k \\ 0 & \text{else} \end{cases}$$

$h_n(X, A) \cong h_n(X/A, pt)$ ← for cell complexes this went via excision

$h_n(D^k, \partial D^k) \cong h_n(S^k)$ because $D^k/\partial D^k \cong S^k$

exact seq of pair

inductively relates homology groups for $(D^k, \partial D^k)$ to S^{k-1}

Additivity ⇝ $h_n(\bigsqcup_\alpha S^k_\alpha, \bigsqcup pt_\alpha) = \bigoplus_\alpha h_n(S^k_\alpha, pt_\alpha) = \begin{cases} \bigoplus_\alpha G & k = n \\ 0 & \text{else} \end{cases}$

$\quad$ ⫽ (S11)

$h_n(\bigvee_\alpha S^k_\alpha, pt)$ ← base computation for cellular homology

To prove theorem: build a theory of cellular $h_n(X)$

by $h_n(X^n, X^{n-1}) \cong G\langle n\text{-cells}\rangle$

exact sequences of pairs defined cellular complex differentials

Differentials are determined by degrees because

gluing maps + quotients coming from cell structure

$f: S^{n-1} \to S^{n-1}$

Homotopy theory: $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$

(based) maps from $S^{n-1} \to S^{n-1}$ are classified up to homotopy by an integer, the degree

Because h_n satisfies homotopy invariance, the cellular diff'l only depend on degrees

$\rightsquigarrow h_n \cong H_n(\ \ ;G)$

Axiom 5 $\quad h_n(pt) = 0 \quad n \neq 0.$

Remove axiom 5, look for h_n satisfying 1 – 4 homotopy, exact seq of pair, excision, additivity

One theory of interest satisfying 1 – 4 not 5:

<u>bordism homology</u>

closed manifold of dim n

$$h_n(X) = \{ f : M^n \to X \} / \sim$$

$f_0 : M_0^n \to X$

$f_1 : M_1^n \to X$

have $f_0 \sim f_1$ if $\exists$ $\underline{N^{n+1}}$ with $\partial N = M_0^n \sqcup \underset{\text{reverse orientation}}{\bar{M}_1^n}$

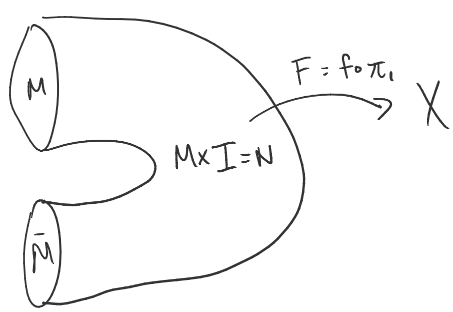
and $F : N^{n+1} \to X$

$F|_{M_0} = f_0$

$F|_{M_1} = f_1$

addition is disjoint union

f and $\bar{f}$ (reverse orient on M) are negatives



$\underline{h_n}(pt) \leftarrow$ cobordism groups of n-manifolds

$$h_n(pt) \leftrightarrow \{ M^n \} / \sim$$

$M_0 \sim M_1 \iff N^{n+1}$ with

$\partial N^{n+1} = M_0 \sqcup \bar{M}_1$