Lecture 24

Friday, March 5, 2021 2:04 PM

Relate homology groups + homotopy groups

$$\pi_1(X, x_0) = \{ f : I \to X \mid f(0) = f(1) = x_0 \} / \sim \quad \leftarrow \text{up to homotopy through such maps}$$

$$= \{ f : S^1 \to X \mid f(1) = x_0 \} / \sim$$

More generally

$$\pi_n(X, x_0) = \{ f : I^n \to X \mid f(x) = x_0 \ \forall x \in \partial I^n \} / \sim$$

$$= \{ f : S^n \to X \mid f(s_0) = x_0 \} / \sim \quad \text{up to homotopy through based maps}$$

fix $s_0 \in S^n$

Define a map Hurewicz homomorphism

$$h_n : \pi_n(X, x_0) \longrightarrow H_n(X) \quad \text{by}$$

$$[f : S^n \to X] \longmapsto f_*(\mathbb{1})$$

Recall $H_n(S^n) \cong \mathbb{Z}$, call generator $\mathbb{1}$

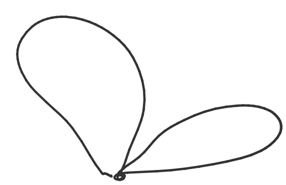
well defined on homotopy classes because if $f \underset{\text{homotopic}}{\simeq} f'$ then $f_* = f'_*$

Group structure on $\pi_n(X, x_0)$

$\underline{n=1}$: $[f]\cdot[g]$

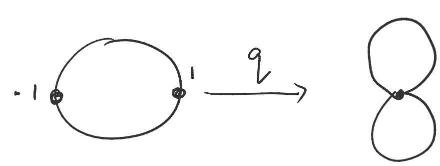
$f : (S^1, 1) \to (X, x_0)$

$g : (S^1, 1) \to (X, x_0)$



$[f]\cdot[g]$ is represented by the map

$$S^1 \xrightarrow[\text{quotient}]{q} S^1 \vee S^1 \xrightarrow{f \vee g} X$$



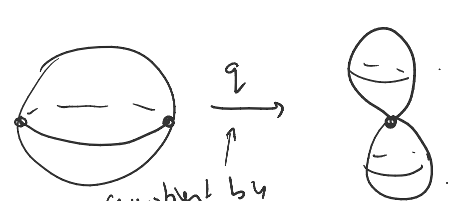
Generally same thing

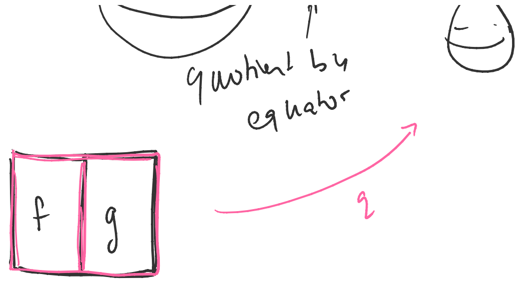
$f : (S^n, s_0) \to (X, x_0)$

$g : (S^n, s_0) \to (X, x_0)$

$[f]\cdot[g]$ is represented by map

$$S^n \xrightarrow{q} S^n \vee S^n \xrightarrow{f \vee g} X$$





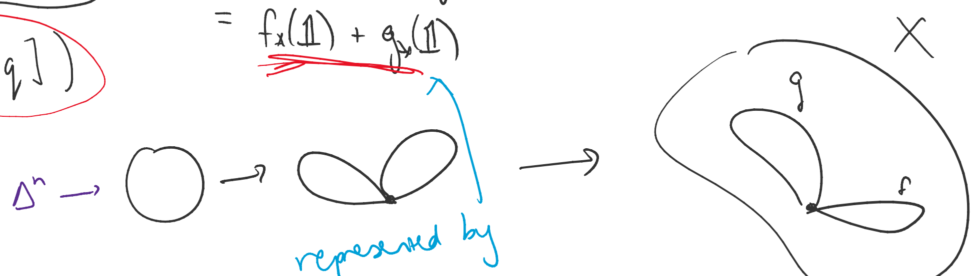
Check h_n is a homomorphism:

$$\underbrace{h_n([f]\bullet[g])}\overset{?}{=} h_n([f]) + h_n([g]) \in H_n(X)$$

$$= \underline{f_*(\mathbb{1}) + g_*(\mathbb{1})}$$

$h_n([(f\vee g)\circ q])$

represented by

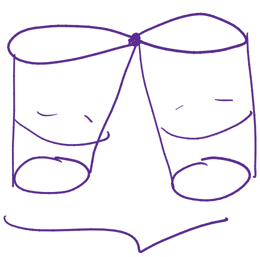


represented by

represents $\mathbb{1} \in H_n(S^n)$ sum of n-cycle

$$\Delta^n \xrightarrow{\sigma_n} S^n \xrightarrow{f \text{ or } g} X$$

quotient by boundary



basic piece: $\underbrace{S^n\times I \vee S^n\times I} \xrightarrow{(f\times id)\vee(g\times id)} X$

express this as sum of simplices

boundary of this n+1 chain will be difference of $h_n([f]\cdot[g])$ & $h_n([f]) + h_n([g])$

Theorem:

① If X is path connected, then the Hurewicz map

$$h_1 : \pi_1(X, x_0) \to H_1(X)$$

is surjective and $\mathrm{Ker}(h_1)$ is the commutator subgroup of $\pi_1(X,x_0)$

i.e. $H_1(X) \cong Ab(\pi_1(X,x_0))$

abelianization

② For $n\geq 2$, if X is $(n-1)$-connected (i.e. $\pi_k(X,x_0)=0$ for $k\leq n-1$)

then $\tilde{H}_k(X) = 0$ for $k=0,\dots,n-1$

and $h_n : \pi_n(X, x_0) \longrightarrow H_n(X)$ is an isomorphism.

---

(1) X path connected

1^{st} show $h_1 : \pi_1(X, x_0) \to H_1(X)$ is surjective.

Given any 1^{st} homology class, it is represented by a 1-cycle

$$\sum \underline{n_i}\, \sigma_i \qquad \sigma_i : \Delta^1 \longrightarrow X$$

s.t. $d_1\left(\sum n_i \sigma_i\right) = 0$

<u>A couple simplifications:</u>

Assume $n_i = \pm 1$ by $n\sigma_i = \underbrace{\sigma_i + \dots + \sigma_i}_{n \text{ times}}$

+ relabeling

$\sigma_i + \sigma_{i+1} + \dots + \sigma_{i+n-1}$

$-\sigma_i \longrightarrow \bar{\sigma}_i \qquad (\bar{\sigma}_i = \sigma_i \circ r)$

reflects Δ^1

$\longleftarrow$ Can assume $n_i = +1$

keeps homology class same

+ d_1 same

$\sigma_i + \bar{\sigma}_i$ is a cycle

$[\sigma_i + \bar{\sigma}_i] = 0$

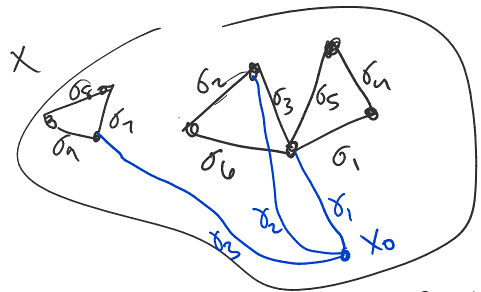
Now my $H_1(X)$ class is repr by $\sigma_1 + \dots + \sigma_N$

+ it is a cycle $\quad d_1(\sigma_i) = \sigma_i^{end} - \sigma_i^{start}$

every end point is start point of some σ_i

$d_1(\sigma_1 + \dots + \sigma_N) = 0$ allows us to pair up end + start points to decompose

$\sigma_1 + \dots + \sigma_N$ into a bunch of loops



$$\underbrace{(\sigma_1+\sigma_4+\sigma_5)}_{\text{loop 1}} + \underbrace{(\sigma_2+\sigma_6+\sigma_3)}_{\text{loop 2}} + \underbrace{(\sigma_7+\sigma_8+\sigma_9)}_{\text{loop 3}}$$

Find $[\sigma_1+\dots+\sigma_N]$ $h_1([\underbrace{\gamma_1(\sigma_1\sigma_4\sigma_5)\gamma_1^{-1}}\cdot\gamma_2(\text{loop }2)\gamma_2^{-1}\dots\gamma_k(\text{loop }k)\gamma_k^{-1}])$

$$\| \qquad\qquad \|$$

$$\underbrace{\gamma_1+\sigma_1+\sigma_4+\sigma_5+\gamma_1^{-1}}+\dots$$

$$[\gamma_1+\gamma_1^{-1}]=0$$

Next: understand $\ker(h_1)$ $\qquad h_1:\pi_1(X,x_0)\to H_1(X)$

Suppose $[f]\in\pi_1(X,x_0)$ has $h_1([f])=0$

$f_*(\mathbb{1})=0$

$\sigma_0:\Delta^1\to S^1$

$\mathbb{1}=[\sigma_0]$ $\qquad \Rightarrow \quad f\circ\sigma_0\in \operatorname{im} d_2$

So $f\circ\sigma_0 = d_2\left(\sum m_j T_j\right)$ $\qquad T_j:\Delta^2\to X$

Similar simplification allows $m_j=\pm 1$

Lets call $d_2T_j = \underbrace{T_j^0 - T_j^1 + T_j^2}_{\text{each is a 1-chain}}$ $\qquad T_j^0 = T_j|_{[\hat{v}_0, v_1, v_2]}$

$T_j^1 = T_j|_{[v_0, v_2]}$

$f\circ\sigma_0 = \sum m_j(T_j^0 - T_j^1 + T_j^2)$ $\qquad T_j^2 = T_j|_{[v_0, v_1]}$