Lecture 25

Monday, March 8, 2021 2:07 PM

Hurewicz Theorem:

$$h_1 : \pi_1(X, x_0) \longrightarrow H_1(X) \qquad \text{Hurewicz homomorphism}$$

$$[f : (S^1, s_0) \to (X, x_0)] \mapsto f_*(\mathbb{1})$$

h_1 is surjective and $\mathrm{Ker}(h_1)$ = commutator subgroup of $\pi_1(X, x_0)$

$\Rightarrow \quad H_1(X) \cong Ab(\pi_1(X, x_0))$

Last time we checked surjectivity.

To look at $\mathrm{Ker}(h_1)$

suppose $\quad f : (S^1, s_0) \to (X, x_0) \quad$ has $\quad f_*(\mathbb{1}) = 0 \in H_1(X)$

$\mathbb{1} \in H_1(S^1) \qquad \sigma_0 : \Delta^1 \to S^1$  $\mathbb{1} = [\sigma_0]$

$f_*(\mathbb{1}) = f \circ \sigma_0 : \Delta^1 \to X$

$f \circ \sigma_0 = d_2\left(\sum m_j T_j\right) \qquad T_j : \Delta^2 \longrightarrow X$

We argued that we can assume that $m_j = \pm 1$

$$d_2(T_j) = T_j^0 - T_j^1 + T_j^2 \quad \leftarrow \text{restrictions to edges}$$

$$f \circ \sigma_0 = \sum_j m_j (T_j^0 - T_j^1 + T_j^2)$$

$$\underline{f \circ \sigma_0} - \sum m_j (T_j^0 - T_j^1 + T_j^2) = 0$$

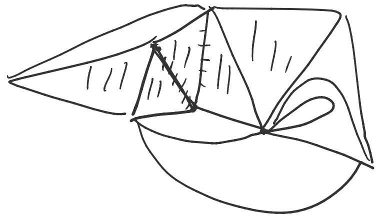
There is some $m_j T_j^k = f \circ \sigma_0$ and all other $m_j T_j^k$'s pair up in cancelling pairs

$$(-1)^k m_j T_j^k + (-1)^{k'} m_{j'} T_{j'}^{k'} = 0$$

Build an abstract surface F from triangles $\Delta_j \cong \Delta^2$

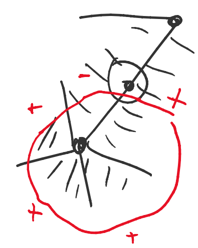
by gluing the edges Δ_i^k with $\Delta_{i'}^{k'}$ according to the pairing

... triangles $\Delta_j = \Delta$
by gluing the edges Δ_j^k with $\Delta_{j'}^{k'}$ according to this pairing.



Builds a surface with one boundary component corresponding to $f \circ \gamma_0$.

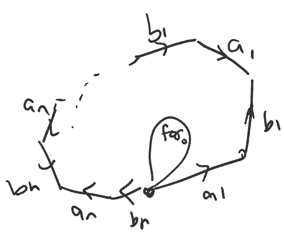
The T_j maps glue together to define a continuous map $T: F \to X$



F is orientable because signs of T_j's match up

Option 1: Use classification of surfaces through polygonal presentations (Lee's Intro to Topological Mflds)

F a surface w/ one boundary component has a polygonal presentation of form.

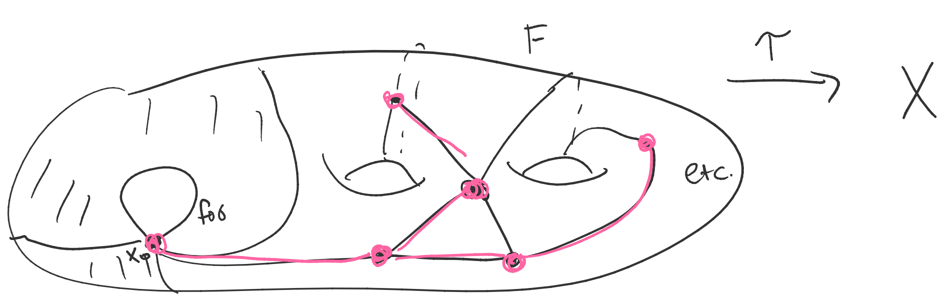


$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$

$\Rightarrow$ $f \circ \gamma_0$ is homotopic to $\underline{a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}}$ $\in$ Commutator subgroup

$[f] \in \pi_1(X, x_0)$ is in the commutator subgroup.

Option 2: Let F have its natural cell structure from Δ_j triangles.



Find a homotopy of T so that all vertices in this triangulation of F to be sent to x_0

Find a homotopy of T so that all vertices in this triangulation of F to be sent to x_0.

Choose a maximal tree in $F' \leftarrow$ 1 skeleton of F

This tree is contractible & has neighborhood U which def retracts onto the tree

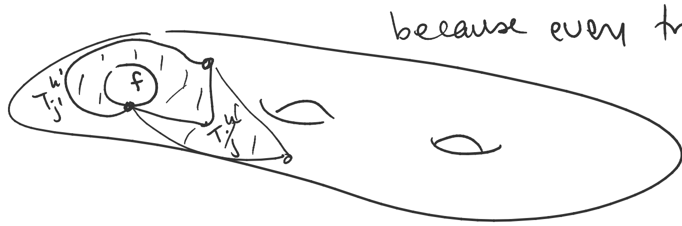
So find a homotopy crushing T(tree) to x_0.

Now call homotoped T, $T' : F \to X$

all edges of T_j's are sent to loops based at x_0 by T'.

Now $f \simeq \prod T_j^{i_k}$ why?

because every triangle is contractible



get $f \simeq T_j^{k} \cdot T_{j'}^{k'}$ (up to inverses)

keep applying more triangles

add in $T_j^0 T_j^{1^{-1}} T_j^2 \longrightarrow$ Eventually cross all triangles, write f as a product of all T_j^k's each edge appears twice once w/ + power + once with − power

In the end f gets written ^up to homotopy as a product of loops which is in commutator subgroup

Anything in commutator subgroup of $\pi_1(X, x_0) \longrightarrow H_1(X)$ $\square$

$c \longmapsto h_1(c)$

$H_1(X)$ is abelian

$c = [a_1, b_1] \cdots [a_n, b_n]$

$h_1(c) = [h_1(a_1), h_1(b_1)] \cdots [h_1(a_n), h_1(b_n)]$

Higher homotopy case: $n \geq 2$

If X is $(n-1)$-connected $(\pi_k(X, x_0) = 0$ for $k = 0, \ldots, n-1)$

then $\tilde{H}_i(X) = 0$ for $i = 0, \ldots, n-1$ and

$h_n : \pi_n(X, x_0) \to H_n(X)$ is an isomorphism.

Outline: (1) 1st assume X is a CW-complex $\leftarrow$ (1a) CW approximation
($\exists X'$ cell complex & $f: X' \to X$ s.t.
$f_* : \pi_k(X', x_0') \to \pi_k(X, x_0)$
isomorphisms $\forall k$)

Such a map $f: X \to Y$
$f_* : \pi_k(X, x_0) \to \pi_k(Y, y_0)$ isom
is called weak homotopy equivalence.

(1b) f weak htpy equiv induces $f_* : H_n(X; G) \to H_n(Y; G)$ isomorphisms.

(2) Use $(n-1)$ connectedness & cellular approx to say that up to weak homotopy equiv
can assume X has a single 0-cell & no cells of dim between 1 & $n-1$
$X^{n-1} = pt$

(3) Compare cellular homology calculation: $\mathbb{Z}\langle n\text{-cells}\rangle / \text{relations imposed by } n+1 \text{ cells}$

to
homotopy group calculation $\pi_n(X, x_0) = \mathbb{Z}\langle n\text{-cells}\rangle / \text{relations from } (n+1)\text{-cells}$

$\pi_i(S^n) \leftarrow$ understand this for $0 \le i \le n$
hard/impossible to calculate in general for $i > n$

$\pi_n(\Sigma X) \cong \pi_{n-1}(X)$