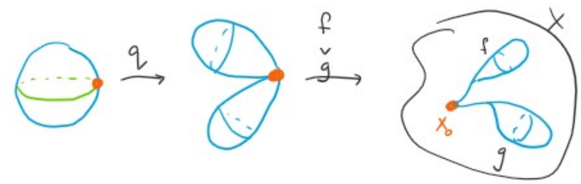
Lecture 26

Wednesday, March 10, 2021 8:46 AM

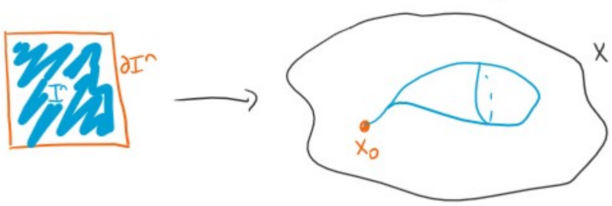
Rapid introduction to higher homotopy groups

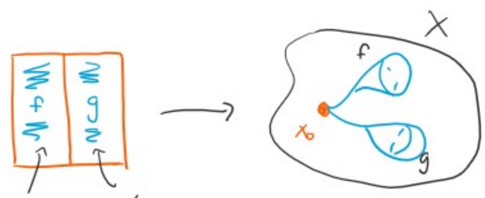
$\pi_n(X, x_0)$:

Defn 1: $\{ f : (S^n, s_0) \to (X, x_0) \} \Big/ \substack{\text{(base point preserving)} \\ \text{homotopy}}$ $\qquad f+g : (S^n, s_0) \to (X, x_0)$



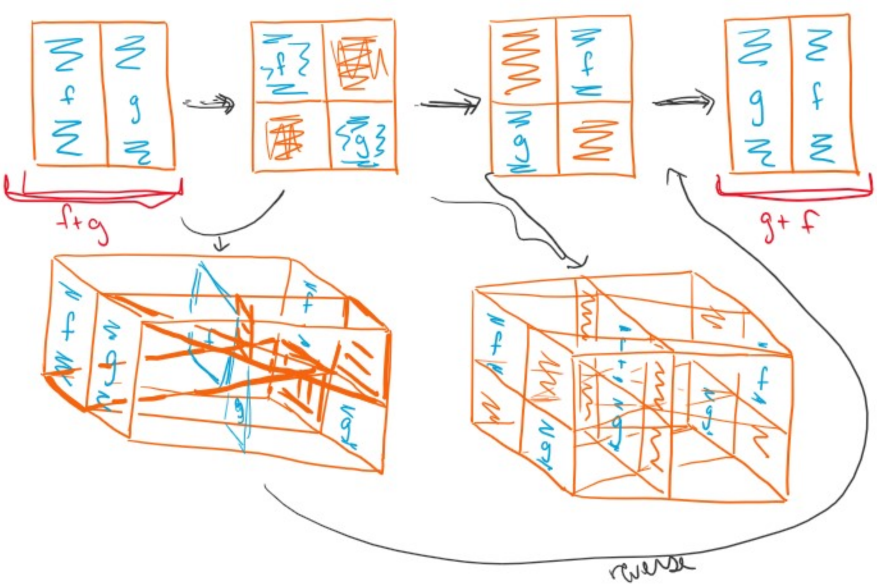
Defn 2: $\{ f : (I^n, \partial I^n) \to (X, x_0) \} \Big/ \substack{\text{homotopy} \\ \text{(through maps of this form)}}$ $\qquad f+g : (I^n, \partial I^n) \to (X, x_0)$





$f(2x_1, x_2, \dots, x_n)$ $\qquad g(2x_1 - 1, x_2, \dots, x_n)$

$\pi_n(X, x_0)$ for $n \geq 2$ is abelian (justifying the "+" notation for group structure)



Homotopy from $f+g$ to $g+f$

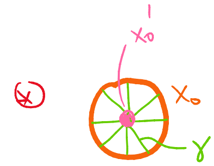
$\alpha : X \to Y \quad \rightsquigarrow \quad \alpha_* : \pi_n(X, x_0) \to \pi_n(Y, y_0) \qquad y_0 = \alpha(x_0)$

$\gamma f \gamma^{-1}$

$\alpha: X \to Y \quad \rightsquigarrow \quad \alpha_*: \pi_n(X, x_0) \to \pi_n(Y, y_0) \qquad y_0 = \alpha(x_0)$

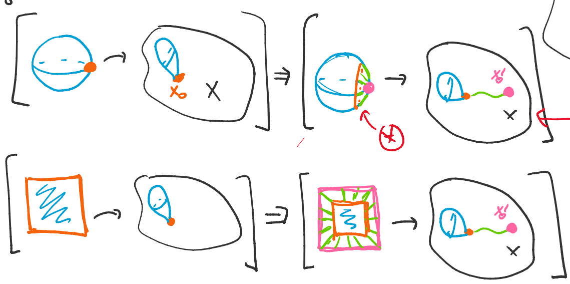
$\qquad\qquad\qquad\qquad [f] \longmapsto [\alpha \circ f]$

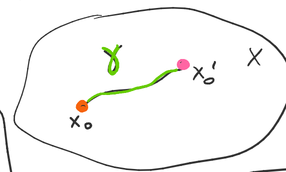
$\gamma f \gamma^{-1}$



If X is path connected (or x_0, x_0' are in same path components)

$$B_\gamma: \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_0')$$



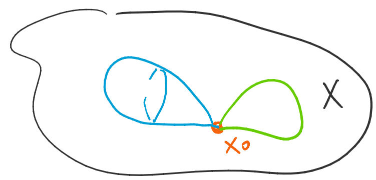


B_γ is not canonical
depends on choice of γ.
Only depends on homotopy
class of γ

For γ a loop $([\gamma] \in \pi_1(X, x_0))$

$$B_\gamma: \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$$

$\rightsquigarrow$ group action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$



Covering maps and higher homotopy groups

Theorem: $p: \tilde{X} \to X$ covering map $\Rightarrow$ $p_*: \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism $(n \geq 2)$

Idea: lifting property

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

$\pi_1(S^n) = 1$ for $n \geq 2$

$\underbrace{f_*(\pi_1(S^n))}_{1} \subset p_*(\pi_1(\tilde{X}))$

Corollary: $\pi_n(S^1) = 0$ for $n \geq 2$. (Since it is covered by $\mathbb{R}$)

Homotopy groups of spheres

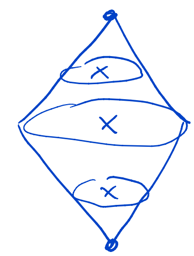
eventually unknown

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}

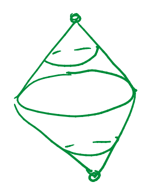
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

Freudenthal Suspension Theorem

Suspension: $X \to \Sigma X = X\times I/\sim$ $\quad (x,0)\sim(x',0)$, $(x,1)\sim(x',1)$



$S^n \to \Sigma S^n \cong S^{n+1}$



$f: S^n \to X \rightsquigarrow \Sigma f: \underset{\substack{\cong \\ S^{n+1}}}{\Sigma S^n} \to \Sigma X \qquad \Sigma f(x,t) = (f(x),t)$

$(f(x),0)\sim(y,0)$
$(f(x),1)\sim(y,1)$

$\Rightarrow \Sigma: \pi_n(X) \to \pi_{n+1}(\Sigma X)$

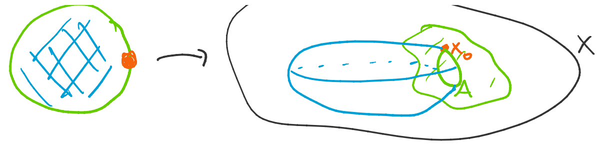
Theorem [Freudenthal] $\Sigma: \pi_n(S^k) \to \pi_{n+1}(S^{k+1})$ is an isomorphism for $n<2k-1$ and surjective for $n=2k-1$.

Relative homotopy groups + exact sequence of pairs

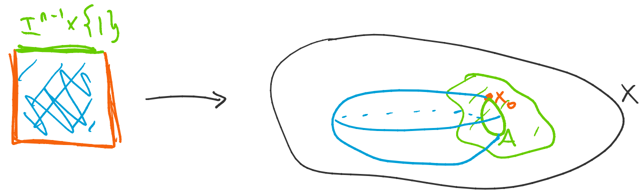
$\pi_n(X,A,x_0) \qquad \{f:(D^n,\partial D^n,d_0)\to(X,A,x_0)\}/$ homotopy through maps of this type

$x_0\in A\subset X$

$x_0 \in A \subset X$



or $\{ f: (I^n, I^{n-1}\times\{1\}, \partial I^n \setminus (I^{n-1}\times\{1\})) \longrightarrow (X, A, x_0) \}$ / homotopy through maps of this type

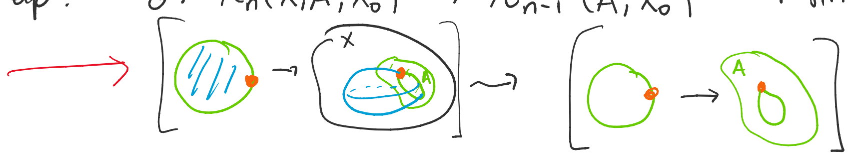


<u>Observe</u>: $\pi_n(X, x_0) = \pi_n(X, \{x_0\}, x_0)$

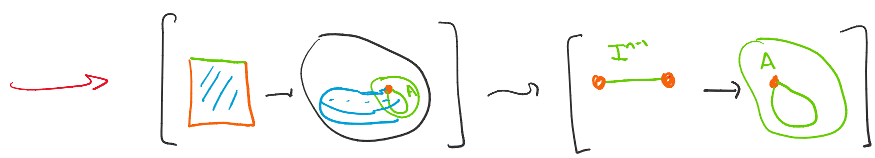
Inclusions of pairs $i: (A, x_0) \hookrightarrow (X, x_0)$ $\quad [(A, x_0, x_0) \to (X, x_0, x_0)]$

$j: (X, x_0, x_0) \hookrightarrow (X, A, x_0)$

Boundary map: $\partial: \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ "restrict to boundary"



n-1 homotopy class in A



<u>Exact Sequence of pairs</u>:

$$\cdots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \cdots$$

Understanding $\pi_n(S^k)$, $\pi_n(D^k, \partial D^k)$ + exact sequence gives some power to study CW complexes

<u>Note</u>: We do not generally understand $\pi_n(S^k)$ for $n \gg k$ & there are limitations on understanding π_n for cell complexes.