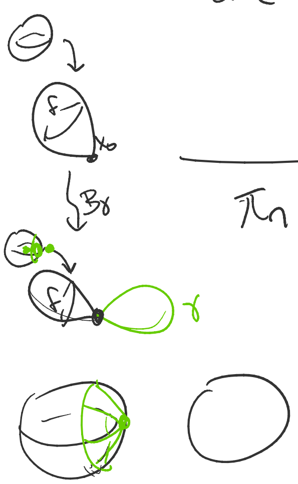
Lecture 27

Friday, March 12, 2021 2:05 PM

Higher homotopy groups and cell complexes



Theorem: Suppose X is a cell complex and Y is obtained from X by attaching an $(n+1)$ cell along a map $f: (S^n, s_0) \to (X^n, x_0)$

The inclusion $i: X \hookrightarrow Y$ induces

$$i_*: \pi_k(X, x_0) \to \pi_k(Y, x_0)$$

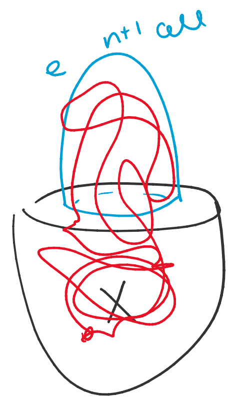
and for $k<n$ i_* is an isomorphism

and for $k=n$ i_* is surjective and $\underline{\mathrm{Ker}(i_*)}$ is generated by $\beta_\gamma[f]$ for all $[\gamma] \in \pi_1(X, x_0)$.

$$\pi_n(Y, x_0) \cong \pi_n(X, x_0)/\mathrm{Ker}(i_*).$$

Proof: Surjectivity $i \leq n$:

For any $[\gamma] \in \pi_i(Y, x_0)$



Initially γ may intersect the $(n+1)$ cell $\gamma: (S^i, s_0) \to (Y, x_0)$

$i < n+1$

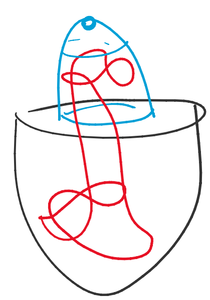
So there is a homotopy of γ such that image of γ misses a point in e^{n+1}

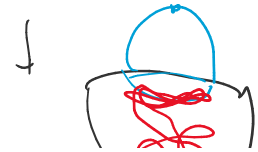
$e^{n+1} - pt$ def retracts to $\partial e^{n+1} \subset X^n$

so can further homotope γ off of e^{n+1}

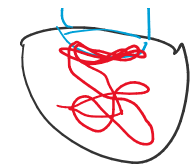
Then image of γ is in X

So $[\gamma] \in \mathrm{im}(i_*)$.





Injectivity for $i<n$: $\gamma: (S^i, s_0) \to (X, x_0)$



Injectivity for $i<n$: $\quad \gamma : (S^i, s_0) \to (X, x_0)$

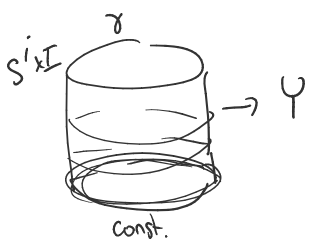
Suppose $i_*[\gamma] = 0$

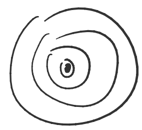
i.e. if γ is homotopic to constant in Y

want to show it is homotopic to constant through a homotopy in X.

γ is homotopic to a constant in Y $\iff$

$$\exists\ \Gamma : D^{i+1} \to Y \quad \text{s.t.} \quad \Gamma|_{\partial D^{i+1}} = \gamma, \quad \partial D^{i+1} = S^i$$

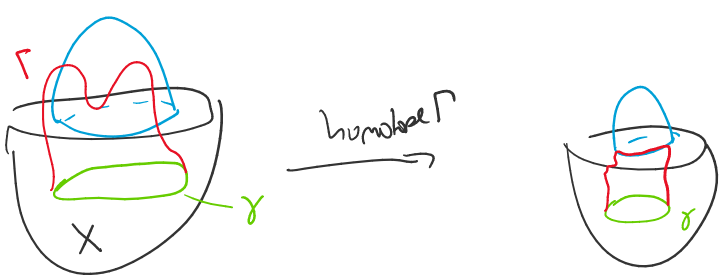




If $i+1 < n+1 \iff i<n$

then can homotope Γ to avoid a point in e^{n+1}

+ then def retract Γ off of $\mathring{e}^{n+1}$, fixing Γ on ∂D^{i+1}



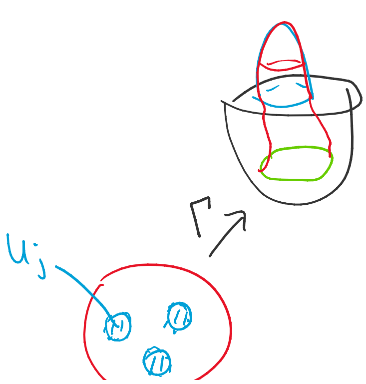
$\Rightarrow$ $\exists$ homotopy in X from γ to constant

$[\gamma] = 1 \in \pi_i(X, x_0)$

$i=n$ case Same idea to start $\quad \gamma : (S^n, s_0) \to (X, x_0)$

$$i_*[\gamma] = 0 \iff \exists\ \Gamma : D^{n+1} \to Y \quad \text{s.t.} \quad \Gamma|_{\partial D^{n+1}} = \gamma, \quad \partial D^{n+1} = S^n$$

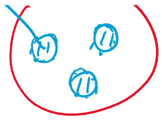
Now try to homotope Γ and it gets stuck on e^{n+1}



Up to homotopy can adjust Γ (fixing ∂D^{n+1}) so that

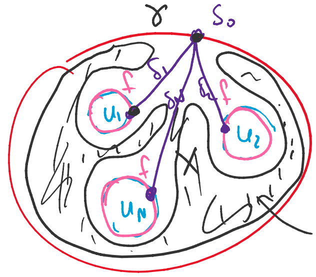
$$\Gamma^{-1}(\mathring{e}^{n+1}) = \coprod_{j=1}^{N} u_j$$

s.t. $\Gamma|_{u_j}$ is a homeomorphism to $\mathring{e}^{n+1}$



D^{n+1}

$\{U_j\}$



f is attaching map for the $n+1$ cell

gives a homotopy from γ to $\prod_{j=1}^{N} B_{\delta_j}[f]$

$\Rightarrow$ $\ker i_*$ is generated by $B_\delta[f]$ ▨

<u>Remark</u>: The theorem said nothing about

$$i_*: \pi_k(X, x_0) \longrightarrow \pi_k(\underset{\text{add } n+1 \text{ cell}}{Y}, x_0) \quad \text{when } k \geq n+1.$$

<u>Defn</u>: A $K(G,n)$ is a topological space X with (G abelian group, $n \geq 1$ ($n=1$ G can be any group))

$$\pi_n(X, x_0) \cong G$$

and $\pi_k(X, x_0) = 0$ for $k \neq n$.

<u>Theorem</u>: For any G, n There exists a cell complex which is a $K(G,n)$.

<u>Proof</u>: 1^{st} choose a presentation for G

$$\langle \{\underset{\text{generators}}{x_\alpha}\}_{\alpha \in A} \mid \{r_\beta\}_{\beta \in B} \rangle$$

Let $X_0 = \bigvee_{\alpha \in A} S^n_\alpha$

$$\pi_n\Big(\bigvee_{\alpha \in A} S^n_\alpha, s_0\Big) \cong \bigoplus_{\alpha \in A} \mathbb{Z}$$

$$\pi_k\Big(\bigvee_{\alpha \in A} S^n_\alpha, s_0\Big) = 0 \quad \text{for } k < n.$$

Don't know for $k > n$.

Attach an $n+1$ cell for each relation $r_\beta \xrightarrow{\text{result}} X_{n+1}$

+ use Theorem to say $\pi_k(X_0, x_0) \cong \pi_k(X_{n+1}, x_0)$ for $k < n$

and $\pi_n(X_{n+1}, x_0) \cong \pi_n(X_0, x_0) / \text{attaching maps} \leftrightarrow r_\beta$

$$\pi_k(X_{n+1}, x_0) = \begin{cases} 0 & k<n \\ G & k=n \\ ? & k>n \end{cases}$$

$\pi_{n+1}(X_{n+1}, x_0)$ may be nontrivial

For each generator of $\pi_{n+1}(X_{n+1}, x_0)$, attach an $(n+2)$-cell imposing a relation killing that generator.

Call result of attaching $(n+2)$-cells for all generators of $\pi_{n+1}(X_{n+1}, x_0)$ X_{n+2}

$$\pi_k(X_{n+2}, x_0) = \begin{cases} 0 & k<n \\ G & k=n \\ 0 & k=n+1 \\ ? & k\geq n+2 \end{cases}$$

Repeat over + over to gradually kill off more + more homotopy groups

X_{n+j} have right homotopy groups up to level $n+j$.

This specifies an infinite dim'l cell complex X which is a $K(G, n)$. $\boxed{\text{A}}$

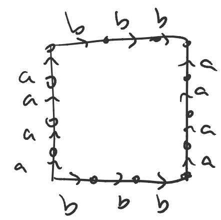
Example: A $K(\mathbb{Z}, 2)$ is given by $\mathbb{C}P^\infty$

Downside of higher homotopy is its to compute + control

Upside: carries alot more info about homotopy type of X

Whitehead Theorem: If $f: X \to Y$ (CW complexes)

Whitehead Theorem: If $f: X \to Y$ (CW complexes) induces isomorphisms $f_*: \pi_n(X) \to \pi_n(Y)$ then f is a homotopy equivalence.



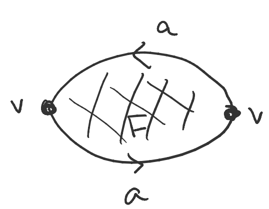
$g \in G \qquad \Phi: G \to \mathrm{Homeo}(X)$

$\Phi(g)(x) =: g \cdot x$

$[\gamma] \; \pi_1(X, x_0)$

$\rightsquigarrow$

$B_\gamma: \pi_n(X, x_0) \to \pi_n(X, x_0)$



$0 \xrightarrow{d_3} \mathbb{Z}/2\mathbb{Z}\langle F \rangle \xrightarrow{d_2 = \times 2} \mathbb{Z}/2\mathbb{Z}\langle a \rangle \xrightarrow{d_1 = 0} \mathbb{Z}/2\mathbb{Z}\langle v \rangle \xrightarrow{d_0} 0$

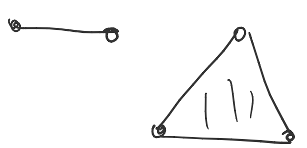
$d_1(a) = v - v$

$d_2(F) = a + a = 2a$

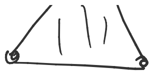
$H_2(\mathbb{RP}^2; \mathbb{Z}/2) = \ker d_2 / \operatorname{im} d_3 \cong \mathbb{Z}/2\mathbb{Z}$

$H_1(\mathbb{RP}^2; \mathbb{Z}/2) = \ker d_1 / \operatorname{im} d_2 = \mathbb{Z}/2\mathbb{Z}$

$H_0(\mathbb{RP}^2; \mathbb{Z}/2) = \ker d_0 / \operatorname{im} d_1 = \mathbb{Z}/2\mathbb{Z}$



$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$



$$d_n(\sigma) = \sum_{i=0} (-1)^i \, \sigma|_{[v_0 \dots \hat{v}_i \dots v_n]}$$

over $\mathbb{Z}/2$ $\quad (-1) = 1$

so $\quad d_n^{\mathbb{Z}_2}(\sigma) = \sum \sigma|_{[v_0 \dots \hat{v}_i \dots v_n]}$

$$d_{n-1} \circ d_n = 0$$