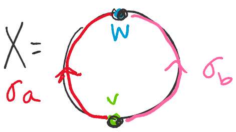
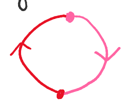
Lecture 3

Friday, January 8, 2021 2:05 PM

Example: $X=$



H_1 gen by $\sigma_a - \sigma_b$



Δ-complex

$\sigma_v : \Delta^0 \to X$ $\quad \sigma_v(v_0) = v$

$\sigma_w : \Delta^0 \to X$ $\quad \sigma_w(v_0) = w$

$$C_1^\Delta(X) = \mathbb{Z}\langle \sigma_a \rangle \oplus \mathbb{Z}\langle \sigma_b \rangle$$
$$C_0^\Delta(X) = \mathbb{Z}\langle \sigma_v \rangle \oplus \mathbb{Z}\langle \sigma_w \rangle$$

$$\underbrace{0}_{C_2^\Delta(X)} \xrightarrow{d_2} C_1^\Delta(X) \xrightarrow{d_1} C_0^\Delta(X) \xrightarrow{d_0} 0$$

$$d_1(\sigma_a) = \sigma_a|_{[(0,1)]} - \sigma_a|_{[(1,0)]} = \sigma_w - \sigma_v$$

Similarly $\quad d_1(\sigma_b) = \sigma_w - \sigma_v$

$d_0(\sigma_v) = 0 \qquad d_0(\sigma_w) = 0$

$H_0^\Delta = \operatorname{Ker} d_0 / \operatorname{im} d_1$

$H_1^\Delta = \operatorname{Ker} d_1 / \operatorname{im} d_2$

d_0 is the 0-map $\quad \operatorname{Ker} d_0 = C_0^\Delta(X) = \mathbb{Z}\langle \sigma_v \rangle \oplus \mathbb{Z}\langle \sigma_w \rangle$

$\operatorname{im}(d_1) =$ all integer combinations of $\underset{\sigma_w - \sigma_v}{\underset{\shortparallel}{d_1(\sigma_a)}}$ and $\underset{\sigma_w - \sigma_v}{\underset{\shortparallel}{d_1(\sigma_b)}}$

$\operatorname{im}(d_1) = \mathbb{Z}\langle \sigma_w - \sigma_v \rangle$

__Ker(d_1):__ $\quad d_1(n\sigma_a + m\sigma_b) = 0$

$\iff n(\sigma_w - \sigma_v) + m(\sigma_w - \sigma_v) = 0$

$\iff (n+m)\sigma_w - (n+m)\sigma_v = 0$

$n = -m$

$\operatorname{Ker}(d_1) = \mathbb{Z}\langle \sigma_a - \sigma_b \rangle$

$\operatorname{im}(d_2) = 0$

__Homology:__ $\quad H_0^\Delta(X) = \operatorname{Ker} d_0 / \operatorname{im} d_1 = (\mathbb{Z}\langle \sigma_v \rangle \oplus \mathbb{Z}\langle \sigma_w \rangle) / \mathbb{Z}\langle \sigma_w - \sigma_v \rangle$

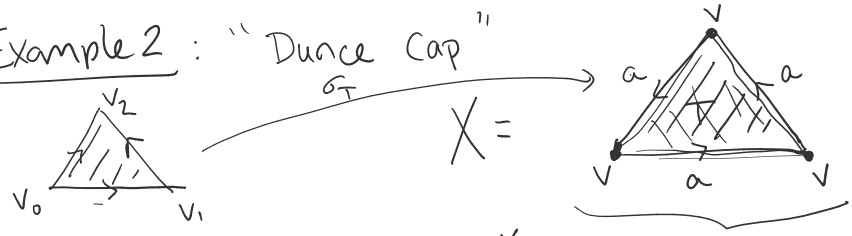
$\mathbb{Z}$-module presented as

$$= \langle \underbrace{\sigma_v, \sigma_w}_{\text{generators}} \mid \underbrace{\sigma_w - \sigma_v = 0}_{\sigma_v + \sigma_w \text{ are same in homology}} \rangle$$

$\cong \mathbb{Z}$ (generated by σ_v)

$H_1^\Delta(X) = \operatorname{Ker} d_1 / \operatorname{im} d_2 = \mathbb{Z}\langle \sigma_a - \sigma_b \rangle / 0 \cong \mathbb{Z}$ generated by $\sigma_a - \sigma_b$

Example 2: "Dunce Cap"



$\sigma_v : \Delta^0 \to X$

$\sigma_a : \Delta^1 \to X$

$\sigma_T : \Delta^2 \to X$

$$0 \xrightarrow{d_3} C_2^\Delta = \mathbb{Z}\langle \sigma_T \rangle \xrightarrow{d_2} C_1^\Delta = \mathbb{Z}\langle \sigma_a \rangle \xrightarrow{d_1} C_0^\Delta = \mathbb{Z}\langle \sigma_v \rangle \xrightarrow{d_0} 0$$

$$d_2(\sigma_T) = \sigma_T\big|_{[v_1, v_2]} - \sigma_T\big|_{[v_0, v_2]} + \sigma_T\big|_{[v_0, v_1]}$$

$$= \sigma_a - (\underset{\substack{\uparrow \\ \text{opposite orientation}}}{-}\sigma_a) + \sigma_a$$

$$= 3\sigma_a$$

$$d_1(\sigma_a) = \sigma_v - \sigma_v = 0$$

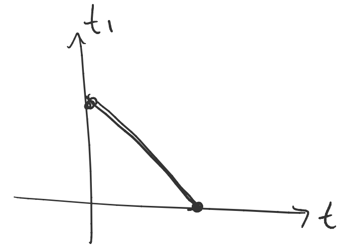
$$d_0(\sigma_v) = 0$$

$$H_0^\Delta(X) = \operatorname{Ker} d_0 / \operatorname{im} d_1 = \mathbb{Z}\langle \sigma_v \rangle / 0 \cong \mathbb{Z}$$

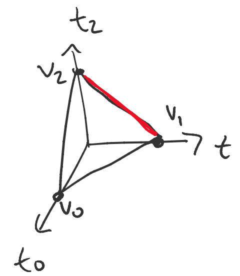
$$H_1^\Delta(X) = \operatorname{Ker} d_1 / \operatorname{im} d_2 = \mathbb{Z}\langle \sigma_a \rangle / \mathbb{Z}\langle 3\sigma_a \rangle \cong \mathbb{Z}_3 \quad \leftarrow \text{torsion!}$$

$$H_2^\Delta(X) = \operatorname{Ker} d_2 / \operatorname{im} d_3 = 0/0 = 0$$

$$\Delta^1 = \{ (t_0, t_1) \in \mathbb{R}^2 \mid \underbrace{t_0 + t_1 = 1}, \ t_0, t_1 \geq 0 \}$$



$$\Delta^2 = \{ (t_0, t_1, t_2) \in \mathbb{R}^3 \mid t_0 + t_1 + t_2 = 1 \quad t_0, t_1, t_2 \geq 0 \}$$



Claim: A face missing $v_0 = (1,0,0)$ is (spanned by v_1, v_2)

$[v_1, v_2]$ $\quad \{ (0, t_1, t_2) \in \mathbb{R}^3 \mid t_1 + t_2 = 1, \ t_1, t_2 \geq 0 \}$

Define a map $\downarrow$

to

Define a map $\downarrow$

$$\Delta^1 = \{ (\tilde{t}_0, \tilde{t}_1) \in \mathbb{R}^2 \mid \tilde{t}_0 + \tilde{t}_1 = 1,\ \tilde{t}_0, \tilde{t}_1 \geq 0 \}$$

want to send $(0,1,0)$ to $(1,0)$

and $(0,0,1)$ to $(0,1)$

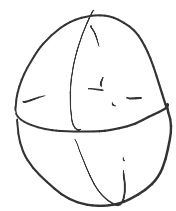
Assume $\psi : \Delta^n \to D^n$ homeomorphism

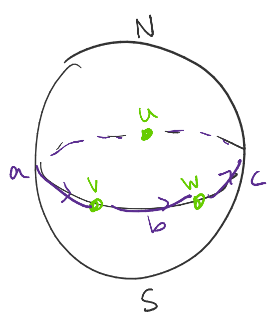
$S^2 = \{ (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1 \}$

Top hemisphere: $\Delta^2 \xrightarrow{\psi} D^2 \xrightarrow{\Phi_N} S^2$

$(u_1, u_2) \longrightarrow (u_1, u_2, \sqrt{1 - u_1^2 - u_2^2})$

$D^2 \xrightarrow{\Phi_S} S^2$





Expect: $u \sim v \sim w$ same in H_0

$a + b + c$ (up to signs)

to be in $\operatorname{Ker} d_1$

and in $\operatorname{im} d_2$