Lecture 4

Monday, January 11, 2021 2:06 PM

Shift HW due date to Fridays
Office Hours on Mondays

Singular Homology

Another homology theory (the homology of another chain complex: $(C(X), d)$)

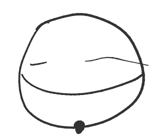
Input: Topological space X (no Δ-complex str needed)

$C_n(X) =$ Free $\mathbb{Z}$ module generated by $\{\sigma : \Delta^n \to X\}$ ← singular chain

σ — any continuous map

"Singular" indicates that σ can be as bad as it wants

In general, there are uncountably infinitely many such generators



Differential $d_n : C_n(X) \to C_{n-1}(X)$

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

Note: Restriction of σ to a face is an $(n-1)$ singular chain

$d_{n-1} \circ d_n = 0$ ← Follows from same calculation in simplicial homology

$H_n(X) = \ker d_n / \operatorname{im} d_{n+1}$ $\qquad C_{n+1}(X) \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X)$

↗ n^{th} singular homology

Not generally practical to compute from the definition.

Eventually we will find that $H_n(X) \cong H_n^{\Delta}(X)$

$H_n(X)$ ↖ for proofs of abstract properties; $H_n^{\Delta}(X)$ ↖ use for calculations

Prop: $H_0(X) \cong \mathbb{Z}^{I}$ ← index set, where I is the index set for the # of path connected components of X.

Proof: $\cdots \to C_1(X) \xrightarrow{d_1} C_0(X) \xrightarrow{d_0} 0$ $\qquad H_0(X) = \ker d_0 / \operatorname{im} d_1$

$C_1(X) = \mathbb{Z}\{\sigma : \Delta^1 \to X\}$ $\qquad C_0(X) = \mathbb{Z}\{\sigma : \Delta^0 \to X\} \cong \mathbb{Z}\{x \in X\}$, $\Delta^0 = \{1\}$ $\qquad \ker(d_0) = \mathbb{Z}\{x \in X\}$

$\operatorname{im} d_1 = \mathbb{Z}\{\ldots\} \ \mathbb{Z}\{x - y \mid x, y \text{ are connected}\}$

$$\operatorname{im} d_1 = \mathbb{Z}\{\underbrace{\sigma|_{v_1}}_{\substack{\text{one endpoint}\\ \text{of path}\\ \sigma}} - \underbrace{\sigma|_{v_0}}_{\substack{\text{other}\\ \text{endpoint}}}\} = \mathbb{Z}\{x-y \mid x,y \text{ are connected by a path in } X\}$$

$\{1\}$

$(x-y=0)$

$$H_0(X) = \mathbb{Z}\langle x \in X \mid x=y \text{ if } x \,\&\, y \text{ are connected by a path}\rangle$$
$$\cong \mathbb{Z}\langle \text{path components of } X\rangle \qquad \square$$

More generally $\underline{H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)}$ if X_α are path components of X

$\underline{\sigma : \Delta^n \to X}$

Variation on singular homology: reduced singular homology

$\tilde{H}(X)$ is homology of reduced chain complex

$\tilde{C}_n(X) = C_n(X)$ when $n \neq -1$

$\tilde{C}_{-1}(X) = \mathbb{Z}$

$\tilde{d}_n = d_n$ when $n \neq 0$

$\tilde{d}_0$ changes

$$C_n(X) \to \cdots \to C_2(X) \to C_1(X) \xrightarrow{\tilde{d}_1} C_0(X) \xrightarrow{\tilde{d}_0} \mathbb{Z} \to 0 \to 0 \cdots$$

$$\tilde{d}_0\Big(\sum_i k_i \sigma_i\Big) = \sum k_i$$

$\sigma_i : \Delta^0 \to X$

$$\tilde{d}_0 \circ \tilde{d}_1(\sigma : \Delta^1 \to X) = \tilde{d}_0(x-y) = 1-1 = 0$$

endpts of σ

If $X \neq \emptyset$,

$\tilde{H}_n(X) \cong H_n(X)$ for $n \neq 0$

$\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$

Reduced homology of a 1 point space is 0 in every degree.

Algebra interlude

Algebra interlude

Defn: Given 2 chain complexes (C, d) (C', d')

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\Phi_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "\Phi_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\Phi_{n-1}"] & C_{n-2} \arrow[r, "d_{n-2}"] \arrow[d, "\Phi_{n-2}"] & \cdots \\
\arrow[r, "d'_{n+2}"] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r, "d'_{n-1}"] & C'_{n-2} \arrow[r, "d'_{n-2}"] &
\end{tikzcd}$$

a chain map $\Phi : (C, d) \to (C', d')$ is a collection of module maps $\Phi_n : C_n \to C'_n$ such that this diagram commutes: $d'_n \circ \Phi_n = \Phi_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$.

Prop: A chain map Φ induces a well-defined map Φ_* on homology

$$H_n = \ker d_n / \operatorname{im} d_{n+1} \qquad H'_n = \ker d'_n / \operatorname{im} d'_{n+1}$$

$\Phi_* : H_n \to H'_n$ is defined by

$\Phi_*([c]) = [\Phi(c)]'$ / $c \in \ker d_n < C_n$

$[\ ] \leftarrow$ equiv class where elts that differ by $\operatorname{im} d_{n+1}$ are equivalent.

$[\ ]' \leftarrow$ " " " $\operatorname{im} d'_{n+1}$ are equiv

Well-defined requires:

① Need $\Phi_n(c) \in \ker d'_n$

By assumption $d_n(c) = 0$

$$d'_n(\Phi_n(c)) = \Phi_{n-1} \circ d_n(c) = \Phi_{n-1}(0) = 0 \checkmark$$

② $\Phi_*[c + d_{n+1}(b)] = [\Phi_n(c) + \Phi_n(d_{n+1}(b))]' = [\Phi_n(c) + \underbrace{d'_n(\Phi_{n+1}(b))}_{\text{is 0 under ' equiv rel}}]'$

$\uparrow$ different representative of $[c] \in H_n$

$\| $

$[\Phi_n(c)]'$

image under Φ_* is same as image of $[c]$

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<u>Moral</u>: chain maps induce maps on homology.