Lecture 5

Wednesday, January 13, 2021 2:07 PM

Algebra: Chain complex

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\Phi_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "\Phi_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\Phi_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "\Phi_{n-2}"] & {} \\
\arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r, "d'_{n-2}"] & C'_{n-2} \arrow[r] & {}
\end{tikzcd}$$

Chain maps Φ

" $\Phi \circ d = d \circ \Phi$ " $\qquad d'_n \circ \Phi_n = \Phi_{n-1} \circ d_n \quad \forall n$

Point: chain maps induce well defined maps on homology

Today: apply to singular homology

From a continuous map $f : X \longrightarrow Y$

$\rightsquigarrow$

chain map $f_{\#} : C(X) \longrightarrow C(Y)$ (sing chain cx)

To define $f_{\#,n} : C_n(X) \longrightarrow C_n(Y)$ $\qquad \sigma_i : \Delta^n \to X$

$C_n(X) = \{\sum k_i \sigma_i\}$ $\qquad$ an elt of $C_n(Y)$

$f_{\#,n}(\sum k_i \sigma_i) := \sum k_i (f \circ \sigma_i)$

$$\begin{tikzcd}
\Delta^n \arrow[r, "\sigma_i"] \arrow[rr, bend right, "f \circ \sigma_i"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Claim: $d \circ f_{\#} = f_{\#} \circ d$

i.e. $d_n^Y \circ f_{\#,n} = f_{\#,n-1} \circ d_n^X$ $\qquad \sigma : \Delta^n \to X$

$$d_n^Y \circ f_{\#,n}(\sigma) = d_n^Y (f \circ \sigma) = \sum_{i=0}^{n} (-1)^i (f \circ \sigma)\Big|_{[v_0 \cdots \hat{v}_i \cdots v_n]} \quad \|$$

$$f_{\#,n-1} \circ d_n^X(\sigma) = f_{\#,n-1}\Big(\sum_{i=0}^{n} (-1)^i \sigma\big|_{[v_0 \cdots \hat{v}_i \cdots v_n]} \Big) = \sum_{i=0}^{n} (-1)^i f \circ \big(\sigma\big|_{[v_0 \cdots \hat{v}_i \cdots v_n]}\big)$$

Restriction to a face is precomposing with inclusion of the face

$F_i : \Delta^{n-1} \to \Delta^n$ which sends Δ^{n-1} to i^{th} face

$f_{\#}$ is post composing with f

Precomposing & post composing commute. $\square$

Since $f_{\#}$ is a chain map, it induces a well-def map on singular homology.

$f_{*,n} : H_n(X) \to H_n(Y)$

Ways that top spaces can be equivalent:

① Homeomorphic : $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad g \circ f = id_X \quad f \circ g = id_Y$

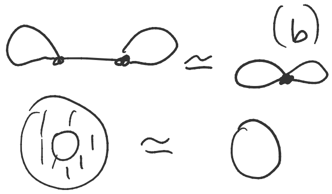
② Homotopy equivalent:

(a) Two continuous maps $f_0, f_1 : X \to Y$ are <u>homotopic</u>

if $\exists$ $F : X \times I \to Y$ s.t. $F(x,0) = f_0(x)$ $\forall x \in X$

$F(x,1) = f_1(x)$

<u>Notation</u>: $f_0 \simeq f_1$



(b) X and Y are homotopy equivalent if $\exists f, g$ s.t.

$X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $g \circ f \simeq id_X \quad f \circ g \simeq id_Y$

<u>Theorem</u>: If $f: X \to Y$ and $g : X \to Y$ are homotopic maps then the induced maps on homology $f_*, g_* : H_n(X) \to H_n(Y)$ are equal.

Goal

<u>Homological algebra interlude</u>:

Chain cxs, chain maps, now chain homotopies

<u>Defn</u>: Given two chain complexes (C, d) (C', d') and two chain maps between them $\Phi_n, \Psi_n : (C_n, d) \to (C'_n, d')$

a <u>chain homotopy</u> between Φ and Ψ is a sequence of maps $\{h_n : C_n \to C'_{n+1}\}_{n \in \mathbb{Z}}$ such that

$$* \quad \boxed{h_{n-1} \circ d_n + d'_{n+1} \circ h_n = \Psi_n - \Phi_n}$$

$$\to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to$$

$\downarrow \Psi_{n+1}, \Phi_{n+1}$ $\quad h_n \quad$ $\downarrow \Psi_n, \Phi_n$ $\quad h_{n-1} \quad$ $\downarrow \Psi_{n-1}, \Phi_{n-1}$

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\Psi_{n+1},\Phi_{n+1}"] & C_n \arrow[r] \arrow[d, "\Psi_n,\Phi_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r] \arrow[d, "\Psi_{n-1},\Phi_{n-1}"] \arrow[dl, "h_{n-1}"'] & {} \\
\arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

" $\Psi_n - \Phi_n = 0$ up to homotopy " ← "after passing to homology"

Prop: If two chain maps are chain homotopic, then their induced maps on homology are equal.

Proof: $\Phi_*, \Psi_* : H_n \to H_n'$ Given $c \in \operatorname{Ker} d_n$, $[c] \in H_n = \operatorname{Ker} d_n / \operatorname{im} d_{n+1}$

$\Phi_*[c] = [\Phi(c)]'$

$\Psi_*[c] = [\Psi(c)]'$ by chain homotopy

$$(\Psi_* - \Phi_*)([c]) = [\Psi(c) - \Phi(c)]' = [\underbrace{h_{n-1} \circ d_n(c)}_{c \in \operatorname{Ker}(d_n)} + d'_{n+1} \circ h_n(c)]'$$

$$= [h_{n-1}(0) + d'_{n+1}(h_n(c))]'$$

$$= [d'_{n+1}(h_n(c))]'$$

$$= [0] \quad \text{since } \uparrow \text{ is in } \operatorname{im}(d'_{n+1}) \qquad \blacksquare$$

Back to topology:

Prop: If $f, g : X \to Y$ are homotopic maps (in topology sense) then the induced singular chain maps $f_\#, g_\# : C(X) \to C(Y)$ are chain homotopic.

Proving this needs topology

Theorem from above is a corollary: $f_*, g_* : H_n(X) \to H_n(Y)$ have $f_* = g_*$ by combining prev. 2 props.

Using theorem (postponing proof of ↑ to Friday), we can get invariance:

Theorem: If X and Y are homotopy equivalent, then $H_n(X) \cong H_n(Y)$ $\forall n$.

↑ isomorphic

Proof:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad g \circ f \simeq id_X \qquad f \circ g \simeq id_Y$$

Get induced maps $\quad f_*: H_n(X) \to H_n(Y)$

$$g_*: H_n(Y) \to H_n(X)$$

$$g_* \circ f_* \underset{\substack{\text{claim} \\ \text{(HW)}}}{=} (g \circ f)_* \underset{\substack{\uparrow \\ \text{by Theorem}}}{=} (id_X)_* \underset{\text{Claim}}{=} id_{H_n(X)}$$

similarly $\quad f_* \circ g_* = id_{H_n(Y)}$.