Lecture 6

Thursday, January 14, 2021 3:47 PM

Main theorem: If X & Y are homotopy equivalent then $H_n(X) \cong H_n(Y)$ sing. homology are isomorphic

Remaining step:

Prop: If $f, g: X \to Y$ are homotopic then

$$f_\#, g_\# : C_n(X) \to C_n(Y) \text{ are chain homotopic.}$$

Proof: $\underbrace{f, g \text{ homotopic}}$ so $\exists$ $F: X \times I \to Y$ s.t.

$$F(x,0) = f(x)$$
$$F(x,1) = g(x)$$

Want to use F to define a chain homotopy between $f_\#, g_\#$ on $C_n(X)$

$$h_n : C_n(X) \to C_{n+1}(Y)$$

$$g_\# - f_\# = h_{n-1} \circ d_n^X + d_{n+1}^Y \circ h_n \qquad \text{looking for such } h_n$$

Proposed definition for h_n: $h_n : C_n(X) \to C_{n+1}(Y)$

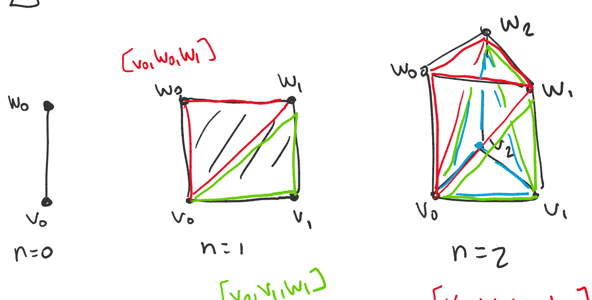
$$h_n(\sigma) := \sum_{i=0}^{n} (-1)^i \underbrace{F \circ (\sigma \times id)\big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}}_{\text{maps from } \Delta^{n+1} \to Y}$$

$\sigma : \Delta^n \to X$

?

$h_n(\sigma)$ is supposed to be an $(n+1)$ chain on Y

$\Delta^n \times I$ can be broken up into $n+1$ simplices as follows:



$[v_0, w_0, w_1, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, v_1, v_2, w_2]$

All ways of writing $\underbrace{[v_0 \dots, v_i, w_i, \dots, w_n]}_{n+2 \text{ vertices spanning } \Delta^{n+1}}$ for any i

$h_n(\sigma) \in C_{n+1}(Y)$

$$\underbrace{\Delta^n \times I} \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$$

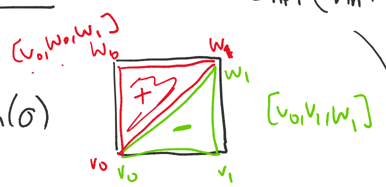
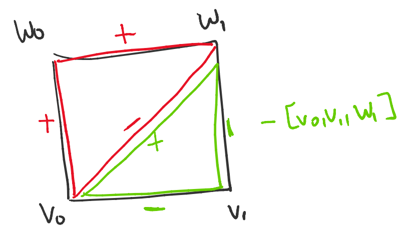
$$h_n(\sigma) = \sum_{i=0}^{n} (-1)^i\, F\circ(\sigma\times id)\Big|_{[v_0 \dots v_i, w_i, \dots, w_n]}$$

Need to check: $g_\# - f_\# = h_{n-1}\circ d_n^X + d_{n+1}^Y \circ h_n$

First rearrange: $d_{n+1}^Y \circ h_n(\sigma) = \overset{\text{top}}{g_\#} - \overset{\text{bottom}}{f_\#} - \overset{\text{sides}}{h_{n-1}(d_n(\sigma))}$

interpret this & match to

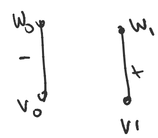
$n=1$ What is $d_{n+1}^Y(h_n(\sigma))$?

$d_{n+1}^Y\Big(h_n(\sigma)$  $\Big) =$ 

$h_n(\sigma) = \sum_{i=0}^{n} (-1)^i\, F\circ(\sigma\times id)\Big|_{[v_0 \dots v_i, w_i, \dots, w_n]}$

$h_{n-1}(d_n(\overline{v_0\, v_1})) = h_{n-1}(\overset{-}{v_0}\quad \overset{+}{v_1}) =$ $\underbrace{F\circ(\sigma\times id)\Big|_{[w_0,w_1]}}_{F(\sigma(\cdot),1)\,=\,g\circ\sigma(\cdot)} - F\circ(\sigma\times id)\Big|_{[v_0,v_1]} - \underbrace{\Big(F\circ(\sigma\times id)\Big|_{[v_1,w_1]} - F\circ(\sigma\times id)\Big|_{[v_0,w_0]}\Big)}_{h_{n-1}(d_n(\sigma))}$

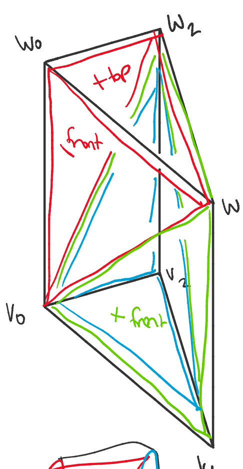
$F\circ(\sigma\times id)\Big|_{[v_0,v_1]} = F(\sigma(\cdot),0) = f\circ\sigma(\cdot)$

$=$ 

$$g_\#(\sigma) - f_\#(\sigma) - \Big(F\circ(\sigma\times id)\Big|_{[v_1,w_1]} - F\circ(\sigma\times id)\Big|_{[v_0,w_0]}\Big)$$

$+[v_0,w_0,w_1,w_2]$

dh_n

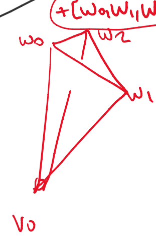


$-[v_0,v_1,w_1,w_2]$

$[v_0,v_1,v_2,w_2]$

$d_{n+1}(h_n(\sigma))$

$+[w_0,w_1,w_2]$ $g_\#$

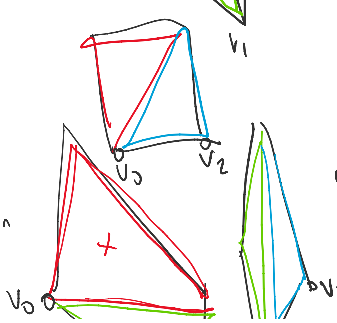


$-[v_0,w_1,w_2] + [v_0,w_1,w_2]$ ← interior face cancels

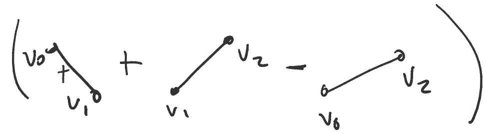
Can check this for all interior faces (Similar to why $d_{n+1}\circ d_n = 0$)

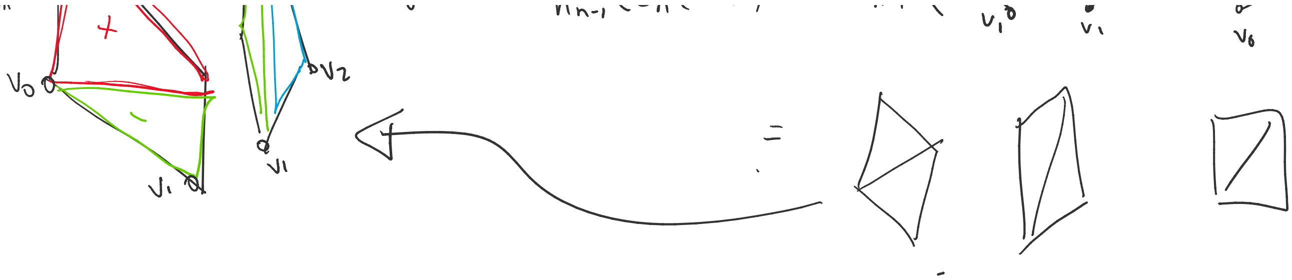
bottom: $-[v_0,v_1,v_2]$ $f_\#$

$h_{n-1}d_n$



Can check signs

$h_{n-1}(d_n(\sigma)) = h_{n-1}\Big($  $\Big)$



V0
V1
V1
V2