Lecture 7

Tuesday, January 19, 2021 3:09 PM

Exact sequences: a collection of modules M_n and maps $f_n : M_n \to M_{n-1}$

$$\cdots \xrightarrow{f_{n+2}} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \xrightarrow{f_{n-2}} \cdots$$

$$\boxed{\text{Condition: } \operatorname{im}(f_{n+1}) = \operatorname{Ker}(f_n) \text{ for all } n}$$

(In particular, it is a chain complex but homology is 0 in every degree)

Examples:

① $0 \xrightarrow{e} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/n \xrightarrow{h} 0$

$f(z) = nz$

$g(z) = z \bmod n$

$0 = \operatorname{im}(e) = \ker(f) \leftarrow f$ is injective

$n\mathbb{Z} = \operatorname{im}(f) = \ker(g)$ ✓

$\mathbb{Z}/n = \operatorname{im}(g) = \ker(h) \leftarrow g$ is surjective

② $\to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \cdots$

$\operatorname{im}(0) = 0 = \operatorname{Ker}(id)$

$\operatorname{im}(id) = \mathbb{Z} = \ker(0)$

Nonexample: $\to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \cdots$ NOT EXACT

$\operatorname{im}(\times n) = n\mathbb{Z} \neq \ker(0) = \mathbb{Z}$

Gathering information about unknown terms in exact sequences:

Example: $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$

f injective: $\operatorname{im} = 0 = \ker(f)$

f surjective: $\operatorname{im}(f) = \ker = B$

f is an isomorphism so $A \cong B$

Defn: Short exact sequence: an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

Properties: f injective, g surjective, $\operatorname{im}(f) = \ker(g)$

Short exact sequence of chain complexes: – sequence of short exact sequences compatible with differential maps

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "i_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[d, "i_n"] & A_{n-1} \arrow[r, "d^A_{n-1}"] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d^B_{n+1}"] \arrow[d, "j_{n+1}"] & B_n \arrow[r, "d^B_n"] \arrow[d, "j_n"] & B_{n-1} \arrow[r, "d^B_{n-1}"] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d] & C_n \arrow[r, "d^C_n"] \arrow[d] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

need squares to commute:

$$i_{n-1} \circ d^A_n = d^B_n \circ i_n \qquad \forall n$$

$$j_{n-1} \circ d^B_n = d^C_n \circ j_n$$

Theorem: A short exact sequence of chain complexes induces a long exact sequence in homology:

$$\cdots \to H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{j_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \to \cdots$$

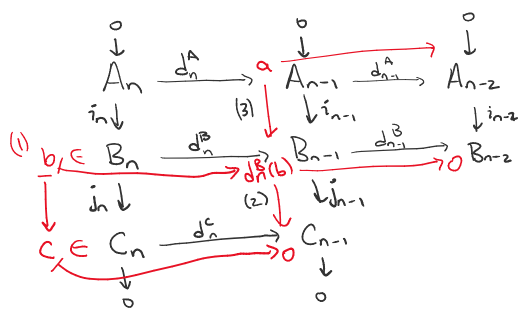
Step I: Define $\partial : H_n(C) \to H_{n-1}(A)$

Step II: Prove exactness ($\mathrm{im} = \mathrm{ker}$) at each position

Step I: $[c] \in H_n(C)$ represented by $c \in \mathrm{Ker}\, d^C_n$

Want $\partial([c]) \in H_{n-1}(A)$ represented by an element of A_{n-1} in $\mathrm{Ker}\, d^A_{n-1}$

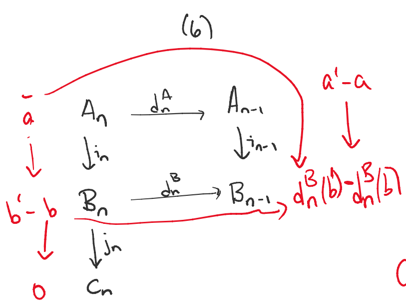
$d^C_n(c)$



	Equations	Justifications
(1)	$j_n(b) = c$	j_n is surjective — Note b may not be unique ☆
(2)	$j_{n-1}(d^B_n(b)) = 0$	$0 = d^C_n(j_n(b)) = j_{n-1}(d^B_n(b))$
(3)	$i_{n-1}(a) = d^B_n(b)$	$\mathrm{im}(i_{n-1}) = \mathrm{Ker}(j_{n-1})$ – note this a is unique (i_{n-1} injective)
(4)	$d^A_{n-1}(a) = 0$	$d^B_{n-1} \circ d^B_n(b) = 0$ $\Rightarrow$ $i_{n-2}(d^A_{n-1}(a)) = 0$ + i_{n-2} is injective; $d^B_{n-1}(i_{n-1}(a)) = i_{n-2}(d^A_{n-1}(a))$

Check independence of choice of b ☆

Suppose $j_n(b') = j_n(b) = c$ and $i_{n-1}(a') = d^B_n(b')$



	Equations	Justifications
(5)	$i_n(\bar{a}) = b' - b$	$b' - b \in \mathrm{Ker}(j_n) = \mathrm{im}(i_n)$ so $\bar{a}$ exists (uniquely)
(6)	$d^B_n(b' - b) = i_{n-1} \circ d^A_n(\bar{a})$	commutativity
(7)	$a' - a = d^A_n(\bar{a})$	$a' - a$ is unique elt mapping to $d^B_n(b' - b)$ by injectivity of i_{n-1}

Conclude: $[a] = [a']$ (they differ by something in $\mathrm{im}(d^A_n)$)

$$\begin{array}{cc} \downarrow & \downarrow{\scriptstyle j_n} \\ 0 & C_n \end{array}$$

Conclude: $[a] = [a']$ (they differ by something in $\mathrm{im}(d_n^A)$)   by injectivity of i_{n-1}

$$\begin{tikzcd}
 & A_n \arrow[r] \arrow[d, "i_n"] & A_{n-1} \arrow[d] & a' - a \arrow[d] \\
\tilde{b} \quad B_{n+1} \arrow[r, "d^B"] \arrow[d, "(8)"'] & B_n \arrow[r] \arrow[d, "j_n"] & B_{n-1} & d_n^B(b') - d_n^B(b) = 0 \\
\tilde{c} \quad C_{n+1} \arrow[r, "d^C"] & C_n & &
\end{tikzcd}$$

$$\tilde{b} \mapsto b' - b \mapsto d_n^B(b') - d_n^B(b), \qquad \tilde{c} \longmapsto c' - c$$

Suppose $c' - c = d_{n+1}^C(\tilde{c})$ ← check answer is indep of choice of representative of homology class $[c] = [c']$

(8)	$j_{n+1}(\tilde{b}) = \tilde{c}$	using j_{n+1} is surjective
(9)	$b' = b + d_{n+1}^B(\tilde{b})$, $j_n(b') = c'$	$b' - b = d_{n+1}^B(\tilde{b})$
(10)	$d_n^B(b') - d_n^B(b) = 0$	$d_n^B(b' - b) = d_n(d_{n+1}(\tilde{b})) = 0$
(11)	$i_{n-1}(a') = d_n^B(b')$, $i_{n-1}(a) = d_n^B(b)$	
(12)	$a' - a = 0$	injectivity of i_{n-1}

Well-defined!

Now have $\partial_n : H_n(C) \to H_{n-1}(A)$.

Next show

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$ is exact in each position:

1. $\mathrm{im}(i_*) = \ker(j_*)$
2. $\mathrm{im}(j_*) = \ker(\partial)$
3. $\mathrm{im}(\partial) = \ker(i_*)$