Lecture 8

Friday, January 22, 2021 2:06 PM

Last time: Trying to prove: Short exact sequence of chain complexes $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_n \arrow[r, "d^A"] \arrow[d, "i_n"'] & A_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d^B"] \arrow[d, "j_n"'] & B_{n-1} \arrow[r] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d^C"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

induces a long exact sequence in homology constructed last time

$$\cdots \longrightarrow H_{n+1}(C) \xrightarrow{\partial} \underset{\uparrow (c)}{H_n(A)} \xrightarrow{i_*} \underset{\Uparrow (a)}{H_n(B)} \xrightarrow{j_*} \underset{\uparrow (b)}{H_n(C)} \underset{\uparrow \text{connecting map}}{\xrightarrow{\partial}} \underset{\uparrow (c)}{H_{n-1}(A)} \longrightarrow \cdots$$

Today: show exactness

(a) $\operatorname{im} i_* = \ker j_*$ (b) $\operatorname{im}(j_*) = \ker \partial$ (c) $\operatorname{im}(\partial) = \ker(i_*)$

$\operatorname{im}(i_*) \subset \ker(j_*)$: $\operatorname{im}(i_*) = i_*([a])$ $[a] \in H_n(A)$

$= [i(a)]$ $\boxed{\operatorname{im}(i) = \ker(j)}$

$j_*([i(a)]) = [j \circ i(a)] = [0] = 0 \in H_n(B)$

so $i_*([a]) \in \ker(j_*)$

$\ker(j_*) \subset \operatorname{im}(i_*)$: Suppose $[b] \in \ker(j_*)$

$[j(b)] = 0 \in H_n(C)$

$\Rightarrow$ $j(b) = d^C_{n+1}(c')$

$$\begin{tikzcd}
b' \arrow[d, mapsto] & B_{n+1} \arrow[r] \arrow[d, "j"'] & B_n \arrow[rr] \arrow[d, "j"] & b \arrow[l] \arrow[d, mapsto] & B_{n-1} \\
c' \arrow[rrr, mapsto] & C_{n+1} \arrow[r, "d^C_{n+1}"'] & C_n & d^C(c') = j(b) &
\end{tikzcd}$$

By exactness j_{n+1} is surjective so $\exists\, b'$ s.t. $j(b') = c'$

Consider $b - d^B(b') \in \ker(j) \overset{\text{exactness}}{=} \operatorname{im}(i)$ ☆

reason: $j(b) - j(d^B(b')) = j(b) - d^C(j(b')) = j(b) - d^C(c') = 0$

☆ $\Longrightarrow$ $b - d^B(b') = i(a)$ for some a ! check $a \in \ker d^A_n$:

$[b] = [b - d^B(b')] = [i(a)] \overset{!}{=} i_*[a]$ justifies

$$\begin{tikzcd}
A_n \arrow[rr] \arrow[d, "i"] & a \arrow[dd, mapsto] \arrow[rrdd, bend left, mapsto] & A_{n-1} \arrow[d, "i"] & d^A_n(a) \arrow[dd, mapsto] & = 0 \\
B_n \arrow[rr, "d^B_n"] & & B_{n-1} & & i \text{ injective} \arrow[u, Rightarrow] \\
 & b - d(b') \arrow[rr] & & 0 &
\end{tikzcd}$$

so $[b] \in \operatorname{im}(i_*)$

$\ker(j_*) = \operatorname{im}(i_*)$

(b) $\operatorname{im}(j_*) = \ker(\partial)$ $H_n(B) \xrightarrow{j_*} \underline{H_n(C)} \xrightarrow{\partial} H_{n-1}(A)$

$\operatorname{im}(j_*) \subset \ker(\partial)$

$\rotatebox{90}{$\in$}$

$j_*([b])$ where $b \in \operatorname{Ker} d_n^B$

$\partial(j_*([b])) = [0]$ ✓

$$\begin{tikzcd}
 & & A_{n-1} \arrow[d] & a \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
b \arrow[d, mapsto] \arrow[rrr, bend right] & B_n \arrow[r, "d_n^B"] \arrow[d, "j"] & B_{n-1} & d^B b \arrow[r, phantom, "="] & 0 \\
j(b) = c \in & C_n & & &
\end{tikzcd}$$

$\operatorname{Ker}(\partial) \subset \operatorname{im}(j_*)$ Suppose $\underline{[c]}$ s.t. $\partial[c] = 0 \in H_{n-1}(A)$

i.e. output of zigzag process applied to c is $d_n^A(\tilde{a})$

$$\begin{tikzcd}
\tilde{a} \arrow[rrr, bend left] & A_n \arrow[r] \arrow[d, "i_n"] & A_{n-1} \arrow[d] & d_n^A\tilde{a} = a \arrow[d] \\
b \arrow[d] \arrow[rrr, bend left] & B_n \arrow[r, "d_n^B"'] \arrow[d, "j_n"] & B_{n-1} & d_n^B b \\
c \in & C_n & &
\end{tikzcd}$$

$\underline{b - i_n(\tilde{a})} \in B_n$

$$d_n^B(b - i_n(\tilde{a})) = d_n^B(b) - d_n^B(i_n(\tilde{a}))$$
$$= d_n^B(b) - i_{n-1} \circ d_n^A(\tilde{a})$$
$$= 0$$

so $[b - i_n(\tilde{a})] \in H_n(B)$ $j_*([b - i_n(\tilde{a})]) = [j(b) - \overset{\substack{0 \text{ by chain exactness} \\ \downarrow}}{j \circ i(\tilde{a})}] = [j(b)] = [c]$

So $[c] \in \operatorname{im}(j_*)$

$\Rightarrow \operatorname{im}(j_*) = \ker(\partial)$

$\underline{\text{Last part}}$: $\operatorname{im}(\partial) = \ker(i_*)$ $H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B)$

$\operatorname{im}(\partial) \subset \ker(i_*)$

$$\begin{tikzcd}
 & & A_{n-1} \arrow[d, "i"] & a \arrow[d] \\
b \arrow[d] \arrow[rrr, bend left] & B_n \arrow[r] \arrow[d] & B_{n-1} & d^B b \\
c & C_n & &
\end{tikzcd}$$

$\partial[c] = [a]$

$i_*([a]) = [i(a)]$
$= [d^B(b)]$
$= 0$.

$\underline{\operatorname{Ker}(i_*) \subset \operatorname{Im}(\partial)}$ Suppose $i_*([a]) = 0 \Rightarrow \overset{\in B_{n-1}}{i(a)} = d_n^B b$

$[a] \in H_{n-1}(A)$

$$\begin{tikzcd}
 & & A_{n-1} \arrow[d] & a \arrow[d] \\
b \arrow[d] \arrow[rr] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[d, "j"'] & d_n^B(b) = i(a) \arrow[d] \\
i(b) \in & C_n \arrow[r] & C_{n-1} \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[rr] \arrow[d] & {} \arrow[d] & {} \arrow[d, "j"'] \arrow[d, "d_n^B(b) ="] & {} \arrow[d, red] \\ j(b) \in C_n \arrow[rr] \arrow[rrr, bend right, red] & & C_{n-1} & {} \end{tikzcd}$$

Claim: $j(b) \in \ker d_n^C$ and so then $\boxed{\partial[j(b)] = [a]}$

$$d_n^C(j(b)) = j_{n-1}(d_n^B(b)) = j_{n-1}(i(a)) \overset{\substack{\text{chain} \\ \text{exactness} \\ \downarrow}}{=} 0$$

$[a] \in \operatorname{im}(\partial)$

▨

Relative homology groups

Given a space X and subspace $A \subset X$ define relative homology of the pair $H_n(X,A)$ as the homology of a chain complex defined as follows:

$$C_n(X,A) = C_n(X) / C_n(A)$$

$C_n(X)$ ← singular simplices map anywhere into X

$C_n(A)$ ← singular simplices mapping fully into A

d_n same as singular – descend to quotient in well-def way because $d_n : C_n(A) \longrightarrow C_{n-1}(A)$

$d_{n-1} \circ d_n = 0$ is same from singular fact.

Natural short exact sequence of chain cxs:

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \longrightarrow 0$$

$$\ker(j) = \operatorname{im}(i)$$

by thm $\Rightarrow$ get longexact seq

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$