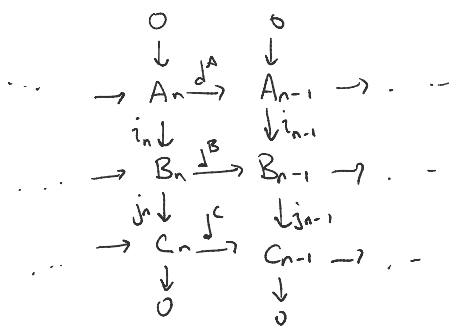


Lecture 8

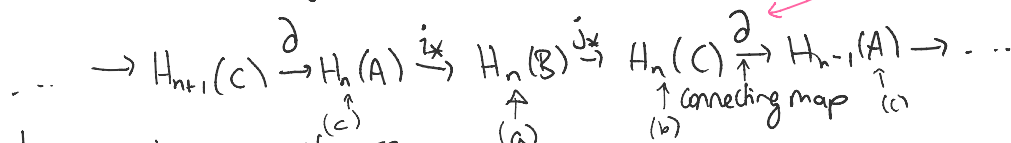
Friday, January 22, 2021 2:06 PM

Trying to prove:
 Last time: Short exact sequence of chain complexes $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$



induces a long exact sequence in homology

constructed last time

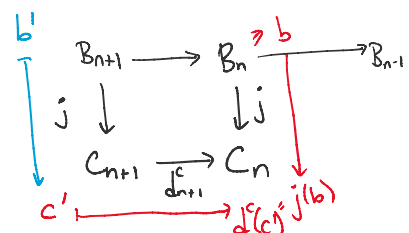


Today: show exactness

- (a) $\text{im } i_* = \text{ker } j_*$
- (b) $\text{im } (j_*) = \text{ker } \partial$
- (c) $\text{im } (\partial) = \text{ker } (i_*)$

$\text{im } (i_*) \subset \text{ker } (j_*)$: $\text{im } (i_*) = i_*([a])$ $[a] \in H_n(A)$
 $= [i(a)]$ $\text{im } (i) = \text{ker } (j)$
 $j_*([i(a)]) = [j \circ i(a)] = [0] = 0 \in H_n(B)$
 so $i_*([a]) \in \text{ker } (j_*)$

$\text{ker } (j_*) \subset \text{im } (i_*)$: Suppose $[b] \in \text{ker } (j_*)$
 $[j(b)] = 0 \in H_n(C)$
 $\Rightarrow j(b) = d_{n+1}^C(c')$



By exactness j_{n+1} is surjective so $\exists b'$ s.t. $j(b') = c'$

Consider $b - d^B(b') \in \text{ker } (j)$ *exactness* $\stackrel{*}{=} \text{im } (i)$

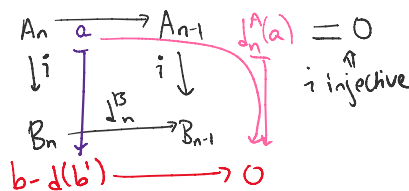
reason: $j(b) - j(d^B(b')) = j(b) - d^C(j(b')) = j(b) - d^C(c') = 0$

$\star \Rightarrow b - d^B(b') = i(a)$ for some a *! check $a \in \text{ker } d_n^A$*

$[b] = [b - d^B(b')] = [i(a)] = i_*[a]$ *justified*

so $[b] \in \text{im } (i_*)$

$\text{ker } (j_*) = \text{im } (i_*)$

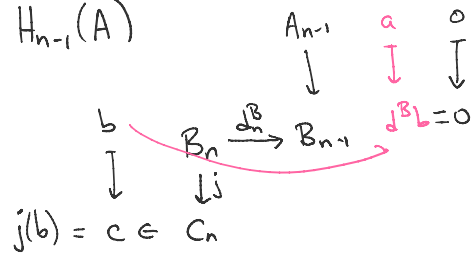


(b) $\text{im}(j_*) = \ker(\partial)$

$$H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A)$$

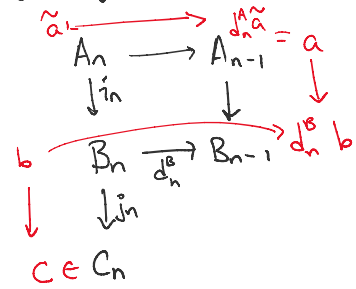
$\text{im}(j_*) \subset \ker(\partial)$

$j_*([b])$ where $b \in \ker d_n^B$
 $\partial(j_*([b])) = [0] \checkmark$



$\ker(\partial) \subset \text{im}(j_*)$ Suppose $[c] \text{ s.t. } \partial[c] = 0 \in H_{n-1}(A)$

i.e. output of zigzag process applied to c is $d_n^A(\tilde{a})$



$b - i_n(\tilde{a}) \in B_n$

$$\begin{aligned} d_n^B(b - i_n(\tilde{a})) &= d_n^B(b) - d_n^B(i_n(\tilde{a})) \\ &= d_n^B(b) - i_{n-1} \circ d_n^A(\tilde{a}) \\ &= 0 \end{aligned}$$

so $[b - i_n(\tilde{a})] \in H_n(B)$

$j_*([b - i_n(\tilde{a})]) = [j(b) - j \circ i(\tilde{a})] = [j(b)] = [c]$

so $[c] \in \text{im}(j_*)$

$\Rightarrow \text{im}(j_*) = \ker(\partial)$

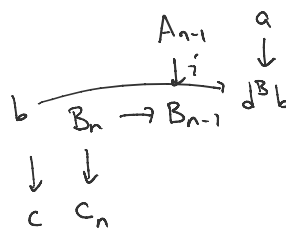
Last part: $\text{im}(\partial) = \ker(i_*)$

$$H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B)$$

$\text{im}(\partial) \subset \ker(i_*)$

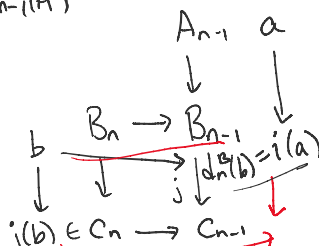
$\partial[c] = [a]$

$$\begin{aligned} i_*([a]) &= [i(a)] \\ &= [d_n^B(b)] \\ &= 0 \end{aligned}$$



$\ker(i_*) \subset \text{im}(\partial)$

Suppose $i_*([a]) = 0 \Rightarrow i(a) = d_n^B(b) \in B_{n-1}$
 $[a] \in H_{n-1}(A)$



$$\begin{array}{ccc} & \xrightarrow{j} & C_{n-1} \\ \downarrow & & \downarrow \\ j(b) \in C_n & \xrightarrow{j} & C_{n-1} \end{array}$$

claim: $j(b) \in \ker d_n^C$ and so then $\partial[j(b)] = [a]$

$d_n^C(j(b)) = j_{n-1}(d_n^B(b)) = j_{n-1}(i(a)) = 0$

$[a] \in \text{im}(\partial)$

chain exactness
 \downarrow
 $= 0$

□

Relative homology groups

Given a space X and subspace $A \subset X$ define relative homology of the pair $H_n(X, A)$ as the homology of a chain complex defined as follows:

$$C_n(X, A) = C_n(X) / C_n(A)$$

\swarrow singular simplices map anywhere into X
 \searrow singular simplices mapping fully into A

d_n same as singular - descend to quotient in well def way because

$$d_n : C_n(A) \rightarrow C_{n-1}(A)$$

$d_{n-1} \circ d_n = 0$ is same from singular fact.

Natural short exact sequence of chain cxs:

$$0 \rightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \rightarrow 0$$

$\ker(j) = \text{im}(i)$

by thm \Rightarrow get long exact seq

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \dots$$