# Lecture 9

Monday, January 25, 2021 2:01 PM

## Relative homology of a pair $(X,A)$ $A \subset X$

Chain complex: $C_n(X)/C_n(A)$ using induced $d_n$ $\xrightarrow{\text{homology}}$ $H_n(X,A)$

$H_n(X,A) = \ker d_n / \operatorname{im} d_{n+1}$

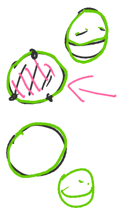
Cycles: i.e. elts in $\ker d_n$ in relative setting

$d_n: C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$

$H_2(X) \cong \mathbb{Z}$

$H_2(X,A) \cong \mathbb{Z}$

$X =$



$A =$

2-simplex is a relative cycle in $(X,A)$ but not a cycle in $X$

$d_n([\Sigma k_i \sigma_i]) = 0 \iff \Sigma k_i d_n \sigma_i$ is an $n-1$ chain in $A$

$[\Sigma k_i \sigma_i]$ Being in $\operatorname{im} d_{n+1}$ in relative setting (a boundary)

i.e. representing the "0" in $H_n(X,A)$ means:

$$[\Sigma k_i \sigma_i] = \underbrace{d_{n+1}(\Sigma n_i \tau_i)}_{} + \underbrace{\Sigma m_i \delta_i}_{}$$

$\tau_i$: $(n+1)$-simplices  $\delta_i$: $n$-simplices in $A$

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A continuous map $f: (X,A) \to (Y,B)$ induces a map on rel homology

$f_*: H_n(X,A) \to H_n(Y,B)$

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## Relative homology as absolute homology

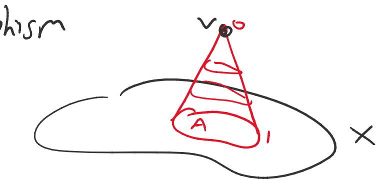
**Theorem 1:** If have a pair $(X,A)$ let $X \cup CA := (X \sqcup (A \times I))/\sim$ (cone of $A$; $I = [0,1]$)

$(a,0) \sim (a',0)$ $\forall a,a' \in A$

$(a,1) \sim a \in X$ $\forall a \in A$

the inclusion $i: (X,A) \hookrightarrow (X \cup CA, CA)$ induces an isomorphism

$i_*: H_n(X,A) \xrightarrow{\cong} H_n(X \cup CA, CA)$




**Comment:** If $X \simeq Y$ and $A \simeq B$ (homotopy equivalence) then $\underbrace{H_n(X,A) \cong H_n(Y,B)}$ (proved on HW)

$$\begin{array}{ccc} C_n(A) & \xrightarrow{i_\#} & C_n(X) \\ \downarrow f'_\# & & \downarrow f_\# \\ C_n(B) & \xrightarrow{i_\#} & C_n(Y) \end{array}$$

compatible in this sense (commuting w/ inclusion)

Then $H_n(X \cup CA, CA) \cong H_n(X \cup CA, v) \cong \tilde{H}_n(X \cup CA)$

the vertex of the cone

on HW homology rel a pt

reduced homology

Under good circumstances $X \cup CA \simeq X/A$ in that case $H_n(X,A) \cong \tilde{H}_n(X/A)$.

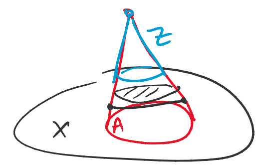
Proving Theorem 1 follows from (result / proof of) following:

<u>Theorem</u> [Excision] Given $Z \subset A \subset X$ s.t. $\bar{Z} \subset \mathring{A}$

The inclusion $i: (X-Z, A-Z) \hookrightarrow (X,A)$ induces an isomorphism

$$i_*: H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X,A)$$

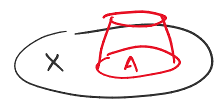
Theorem 1 applies excision for $\underbrace{\text{open cone}}_{A\times[0,\frac{1}{2}]} \subset CA \subset X \cup CA$




$(X\cup CA, CA)$

$(\underbrace{X \cup CA \setminus Z}, CA \setminus Z)$

$\simeq (X, A)$




Key lemma to prove excision: "refinement lemma"

Suppose $\mathcal{U} = \{U_j\}$ is an open cover of $X$

Define $C_n^{\mathcal{U}}(X) \leftarrow$ generated by the singular n-simplices that have image completely contained in some $U_j$ from the cover


Can use same diff'l as usual $d_n$

$$C_n^{\mathcal{U}}(X) \xhookrightarrow{\text{inclusion}} C_n(X)$$ induces an isomorphism on homology

$$H_n^{\mathcal{U}}(X) \cong H_n(X) \quad \forall n.$$

$(X-Z, A-Z) \hookrightarrow (X,A)$ want to pick a cover: $\{\underbrace{X-\bar{Z}}, \mathring{A}\}$ $(\bar{Z} \subset \mathring{A})$

Common use of relative homology in manifold topology: Say $M$ is a m dim'l mfld with boundary $\partial M$

$$H_n(M, \partial M) \cong H^{m-n}(M)$$

(///) $\uparrow$ a version of Poincaré duality Poincaré-Lefschetz

In 215C Poincaré duality: If $M$ is a mfld of dim $m$ w/ no boundary

PD: $H_n(M) \cong H^{m-n}(M)$

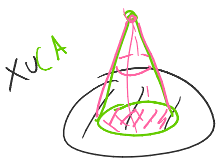
$\uparrow$ homology $\qquad \uparrow$ cohomology

$$H_n(M) \cong H^{m-n}(M, \partial M)$$

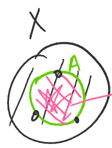
Exact sequence $\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$

is useful to calculate

$\tilde{H}_n(X) = 0$




$H_2(X \cup CA) \cong \mathbb{Z}$




2-chain $\sigma$

relative                      absolute

$\sigma \in \operatorname{Ker} d_2$ even though not in $\ker d_2$

$[\sigma] \neq 0$ in $H_2(X,A) \cong \mathbb{Z}$