HOMEWORK 1 (MAT 215B) TOPOLOGY

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Every solution should contain complete rigorous proofs, verifying that your answer satisfies the properties required.

(1) The n-simplex was defined using barycentric coordinates as

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geqslant 0\, \forall i\}.$$

We denote by v_i the vertex of the n-simplex where $t_i = 1$ and all other coordinates are 0. The notation $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ refers to the face of the n-simplex spanned by all the vertices *except* v_i.

To define the definition of a Δ-complex, we implicitly need an identification of each face of the n-simplex with the standard $(n-1)$-simplex. Your job for this problem is to work out that identification explicitly in coordinates.

Find an explicit map in terms of the barycentric coordinates t_j which sends the i^{th} face $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ of Δ^n to the standard $(n-1)$ simplex $\Delta^{n-1} \subset \mathbb{R}^n$ such that the vertices $v_0, \dots, \widehat{v_i}, \dots, v_n$ are sent to the standard vertices $w_0, \dots, w_{n-1}$ of Δ^{n-1} in an order preserving way.

[Tip: Work it out first for the $n = 2$ case and potentially some other low dimensional examples, and then look for the general pattern.]

(2) Describe a Δ-complex structure on the n-dimensional sphere

$$S^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

You may assume without writing out an explicit equation, that there is a homeomorphism from the n-simplex to the closed n-disk

$$D^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leqslant 1\}.$$

Make sure to verify that your Δ-complex structure satisfies all the properties required by the definition of a Δ-complex.

(3) Describe a Δ-complex structure on the genus g orientable surface. You may use the identification of the orientable genus g surface with the polygonal representation given by a $4g$-gon where sides are identified according to the word $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$ (see e.g. Hatcher page 5). (Verify it satisfies the required properties.)

(4) Consider the following family of Δ-complex structures on the circle S^1. Here it will be convenient to identify the topological space S^1 as $\mathbb{R}/\mathbb{Z}$, or equivalently, the interval $[0, 1]$ with the endpoints 0 and 1 identified.

$$S^1 = [0, 1]/\sim \qquad 0 \sim 1.$$

For each integer $n \geqslant 1$, we get a Δ-complex structure on S^1 with maps $\sigma_{0,0}, \dots, \sigma_{0,n-1} : \Delta^0 \to S^1$ and $\sigma_{1,0}, \dots, \sigma_{1,n-1} : \Delta^1 \to S^1$. The first set of maps are defined by

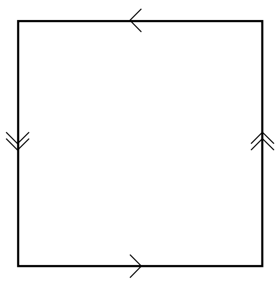
$$\sigma_{0,k}(v_0) = k/n$$

for the unique point $v_0 \in \Delta^0$. For $(t_0, t_1) \in \Delta^1$ where $t_0 + t_1 = 1$ (i.e. $t_1 = 1 - t_0$) and $t_0, t_1 \geqslant 0$ (i.e. $0 \leqslant t_0 \leqslant 1$), define

$$\sigma_{1,k}((t_0, t_1)) = \frac{t_0 + k}{n}.$$

(a) Verify that for each n, this satisfies the requirements for a Δ-complex.
(b) Determine the simplicial chain complex and differential boundary maps for each Δ-complex structure.
(c) Calculate the simplicial homology for each Δ-complex structure.
(d) Verify that the isomorphism type of the homology groups are the same regardless of which of these Δ structures you use (even if the chain complex and differentials differ).

(5) Use the Δ-complex structure from problem (2) to calculate the simplicial homology of S^n.
(6) Use the Δ-complex structure from problem (3) to calculate the simplicial homology of the genus g orientable surface.
(7) One way to represent the real projective plane is from the following polygonal identification where sides are identified as indicated by the arrows and double arrows.



Define a Δ-complex structure on this space, determine the corresponding chain complex and differential boundary maps, and prove that the homology groups in dimensions $0, 1, 2$ are isomorphic to $\mathbb{Z}$, $\mathbb{Z}_2$, 0 respectively.

(8) (See Hatcher Section 2.1 Exercises Problem 8) Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \ldots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as shown in the figure of Hatcher p. 131, so that each T_i shares a common vertical face with its two neighbors, T_{i-1} and T_{i+1} (subscripts taken mod n). Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are isomorphic to $\mathbb{Z}$, $\mathbb{Z}_n$, 0, $\mathbb{Z}$ respectively. (Such X are examples of lens spaces.)