HOMEWORK 2 (MAT 215B) TOPOLOGY

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These exercises study fundamental examples and properties of singular homology, and practice homological algebra calculations.

Every solution should contain complete rigorous proofs, verifying that your answer satisfies the properties required.

(1) Calculate and give a proof in your own words, using only the definition of singular homology, the values of $H_n(X)$ when X is a one point space.

(2) Let X be a topological space, and $x_0 \in X$ a point in X. Consider the singular n-chain $\sigma : \Delta^n \to X$ defined by $\sigma(\mathbf{t}) = x_0$ for all $\mathbf{t} \in \Delta^n$ (the constant map).

   (a) Prove that σ is in the kernel of d_n and thus represents a homology class $[\sigma] \in H_n(X)$ if and only if n is odd.

   (b) Prove that when n is odd, σ is in the image of d_{n+1} and thus $[\sigma] = 0 \in H_n(X)$.

(3) This exercise is purely algebraic. Recall that a chain complex is any sequence of modules C_n with maps

$$d_n : C_n \to C_{n-1}$$

such that $d_{n-1} \circ d_n = 0$.

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

From any such given chain complex, define a new chain complex by $\widetilde{C}_n = C_n \oplus C_{n+1}$ with

$$\widetilde{d}_n : \widetilde{C}_n = C_n \oplus C_{n+1} \to \widetilde{C}_{n-1} = C_{n-1} \oplus C_n$$

defined by $\widetilde{d}_n(a, b) = (-d_n(a), d_{n+1}(b) + a)$. Prove that this is actually a chain complex and that the homology of this complex, $\widetilde{H}_n = \ker(\widetilde{d}_n)/\operatorname{im}(\widetilde{d}_{n+1})$ satisfies $\widetilde{H}_n = 0$ for all n. A chain complex with this property that the homology is 0 in every degree is called *acyclic*.

(4) This exercise is also purely algebraic. Recall that given two chain complexes, C and C', a *chain map* $\Phi : C \to C'$ is a sequence of maps $\Phi_n : C_n \to C'_n$ such that $\Phi_{n-1} \circ d_n = d'_n \circ \Phi_n$ for all n.

   We a new complex $C(\Phi)$ by:

$$C(\Phi)_n = C_n \oplus C'_{n+1}$$

and

$$d^\Phi_n(c_n, c'_{n+1}) = (-d_n(c_n), \Phi_n(c_n) + d'_{n+1}(c'_{n+1}))$$

   (a) Prove that $(C(\Phi), d^\Phi)$ is a chain complex.

   (b) If Φ is a chain map whose induced map on homology is a graded isomorphism (i.e. $\Phi_{*n} : H_n \to H'_n$ is an isomorphism for all n), prove that the homology of the mapping cone complex is 0 in every degree.

   (c) Give a quick reproof of problem (3) by realizing it as a special case of such a complex $C(\Phi)$–what is the chain map Φ which induces an isomorphism on homology?

(5) In this exercise, you'll determine some fundamental examples and properties of the maps induced on homology from continuous maps between spaces. We use the notation that for a continuous map $f : X \to Y$, the induced map on homology is denoted by $f_* : H_n(X) \to H_n(Y)$. (See class notes or Hatcher to recall precisely how f_* is defined in terms of f.)

(a) Let X be a topological space, and $id_X : X \to X$ the identity map on X. Prove that $(id_X)_* : H_n(X) \to H_n(X)$ is the identity map on $H_n(X)$.

(b) Let X, Y, Z be topological spaces and $f : X \to Y$ and $g : Y \to Z$ continuous maps. Prove that $(g \circ f)_* = g_* \circ f_*$.

(c) Suppose X and Y are any topological spaces, $y_0 \in Y$ is a chosen point, and $f : X \to Y$ is the constant map $f(x) = y_0$ for all $x \in X$. What does the map $f_* : H_n(X) \to H_n(Y)$ do for each value of n?