HOMEWORK 3 (MAT 215B) TOPOLOGY

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This assignment is about exact sequences and the relative homology of a pair.

(1) Fill in the unknowns to make the following sequences exact.
   (a) Determine the module A, the map f and the map g to make the following exact:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(id,0)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

   (b) Determine the module B to make the following sequence exact

$$0 \longrightarrow B \longrightarrow 0$$

   (c) Determine the modules A_n (up to isomorphism) and the maps f_n and g_n to make the following sequence exact:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{f_1} A_1 \xrightarrow{g_1} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{f_2} A_2 \xrightarrow{g_2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{f_n} A_3 \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{f_n} A_n \xrightarrow{g_n} \mathbb{Z} \xrightarrow{\times n+1} \mathbb{Z} \xrightarrow{f_{n+1}} A_{n+1} \xrightarrow{g_{n+1}} \cdots$$

(2) For each of the following, find two non-isomorphic ways to fill in the unknowns making the sequence exact:
   (a) Find two non-isomorphic modules which could serve as A such that you can define maps to make the following sequence exact:

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

   (b) Find at least two different possibilities for B and C such that you can define maps to make the following sequence exact:

$$0 \longrightarrow B \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow C \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

   (Can you find and prove you have found all possibilities for the isomorphism types of B and C)?

(3) Using the long exact sequence for the homology of a pair, prove that if x_0 is a point in X then the relative homology of the pair $(X, \{x_0\})$ is isomorphic to the reduced homology of X:

$$H_n(X, \{x_0\}) \cong \tilde{H}_n(X)$$

(4) This problem is about maps between exact sequences and involves lots of diagram chasing.

(a) Suppose we have the following commutative diagram such that the horizontal rows are exact, ϕ_1 is surjective, and ϕ_2 and ϕ_4 are injective.

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 \end{array}$$

Show that ϕ_3 is injective.

(b) Suppose we have the following commutative diagram such that the horizontal rows are exact, ϕ_2 and ϕ_4 are surjective, and ϕ_5 is injective.

$$\begin{array}{ccccccc} A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} \\ B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

Show that ϕ_3 is surjective.

(c) Combine the above two cases to conclude the five lemma: Suppose the following diagram is commutative, the horizontal rows are exact, and $\phi_1, \phi_2, \phi_4, \phi_5$ are isomorphisms.

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

Show that ϕ_3 is an isomorphism.

(5) Using the long exact sequence of the homology of a pair, together with the five lemma from the previous problem to prove the following. Suppose there is a map $f : X \to Y$, $A \subseteq X$, $B \subseteq Y$ and $f(A) \subseteq B$, such that f induces a homotopy equivalence from X to Y and its restriction induces a homotopy equivalence from A to B. Prove that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism.

(6) This is a different way to solve 4(b) from last week's assignment. Recall that if we are given a chain map $\Phi_n : C_n \to C'_n$ such that $\Phi_{n-1} \circ d_n = d'_n \circ \Phi_n$ for all n, we can define a new complex $C(\Phi)$ by:

$$C(\Phi)_n = C_n \oplus C'_{n+1}$$

and

$$d_n^{\Phi}(c_n, c'_{n+1}) = (-d_n(c_n), \Phi_n(c_n) + d'_{n+1}(c'_{n+1}))$$

The "shifted" chain complex $C'[1]$ is the same as C' but with the indexing shifted by 1: $(C'[1])_n = C'_{n+1}$.

(a) Show that the map $i_n : (C'[1])_n = C'_{n+1} \to C(\Phi) = C_n \oplus C'_{n+1}$ defined by $i_n(b) = (0, b)$ is a chain map from $C'[1]$ to $C(\Phi)$.

(b) Show that the map $p_n : C(\Phi) = C_n \oplus C'_{n+1} \to C_n$ defined by $p_n(a, b) = (-1)^n a$ is a chain map from $C(\Phi)$ to C.

(c) Show that $0 \to C'[1] \to C(\Phi) \to C \to 0$ is a short exact sequence of chain complexes. Deduce that we have the following long exact sequence of homologies:

$$\cdots \xrightarrow{\partial} H'_{n+1} \xrightarrow{i_*} H_n(\Phi) \xrightarrow{p_*} H_n \xrightarrow{\partial} H'_n \xrightarrow{i_*} H_{n-1}(\Phi) \xrightarrow{p_*} H'_{n-1} \xrightarrow{\partial} \cdots$$

(d) Show that the connecting homomorphism ∂ is equal to the map Φ_* using the definition of the connecting map ∂ from the construction of the long exact sequence on homology from the short exact sequence of chain complexes.

(e) Using exactness of the sequence, show that if Φ_* is an isomorphism at each level, then $H_n(\Phi) = 0$ for all n.