HOMEWORK 4 (MAT 215B) TOPOLOGY

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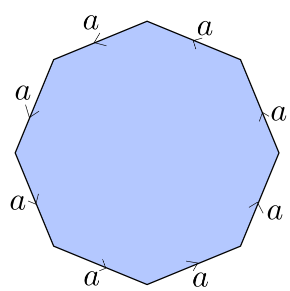
(1) Using the isomorphism between simplicial and singular homology and a Δ-complex structure of your choice, calculate $H_n(X)$ where X is the space obtained from identifying all n sides of an n-gon in a counter-clockwise orientation as in the figure below.



(2) In this problem, we will reprove the calculation of homology for the genus g surface.

(a) Put a Δ-complex structure on the torus T^2, and provide the complete computation of the simplicial homology groups of T^2 using this Δ-complex structure. Using the isomorphism between simplicial and singular homology, conclude the calculation of the isomorphism types of the singular homology groups of the torus.

(b) Let $D' \subset T^2$ be a small closed disk embedded in the torus. Calculate the isomorphism type of $H_n(T^2, D')$ for all n. Let $D \subset D' \subset T^2$ be a smaller radius open disk such that $\overline{D}$ is contained in the interior of D'. Calculate $H_n(T^2 \backslash D, D' \backslash D)$ for all n, and specify cycles representing the generators.

(c) Let Σ_2 be the connected sum of two copies of T^2: namely $\Sigma_2 = \left((T_1^2 \backslash D_1) \sqcup (T_1^2 \backslash D_1)\right) / \sim$ where the equivalence relation glues together the boundaries of the two copies of D along the identity map. Let A denote the subset $A = ((D_1' \backslash D_1) \sqcup (D_2' \backslash D_2)) / \sim$. Prove that the quotient map induces an isomorphism on relative homology

$$q_* : H_n\left(T_1^2 \backslash D_1 \sqcup T_2^2 \backslash D_2, D_1' \backslash D_1 \sqcup D_2' \backslash D_2\right) \to H_n(\Sigma_2, A)$$

Use this to calculate

$$H_n(\Sigma_2, A)$$

for each n, and specify cycles representing the generators.

(d) Show that A is homotopy equivalent to S^1, and use this to calculate $H_n(A)$ for each n.

(e) Calculate the map induced by the inclusion

$$i_* : H_n(A) \to H_n(\Sigma_2)$$

by specifying where there generators are sent.

(f) Using the exact sequence of the pair (Σ_2, A) and your previous results, calculate $H_n(\Sigma_2)$ up to isomorphism for all n.

(g) Generalize this argument for the connected sum of g copies of T^2 (you can either choose $g-1$ small disks on the first copy of T^2 to delete and glue on $g-1$ additional $T^2 \backslash D$'s, or you can inductively connect sum Σ_{g-1} with T^2).

(3) Let T^2 denote the torus. Using the long exact sequence of a pair, calculate relative homology for the following pairs, and describe representatives for a generating set. Note that you will need to understand the maps induced by the inclusion of the subspace into the larger space.
(a) (T^2, M) where M is a meridional circle.
(b) (T^2, C) where C is a circle in T^2 which bounds a disk.
(c) $(T^2, M \cup M')$ where M and M' are disjoint parallel meridional circles.

(4) A *knot* is the image of an embedding of S^1 into S^3. Let $K \subset S^3$ be a knot, such that it has a closed neighborhood N homeomorphic to $S^1 \times D^2$ where D^2 denotes the closed disk. Let ∂N denote the boundary of the neighborhood N (i.e. the image of $S^1 \times S^1$ under the identification of N with $S^1 \times D^2$). Using the exact sequence of the pair (S^3, N) and excision, calculate the relative homology groups of the knot complement rel boundary: $H_n(S^3 \backslash \mathring{N}, \partial N)$ for all n.