HOMEWORK 5 (MAT 215B) TOPOLOGY

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These problems are about cellular homology and degree.

(1) This problem is about Euler characteristic.

  (a) Let (C_n, d_n) be any chain complex of modules over $\mathbb{Z}$ such that all but finitely many C_n are 0 and all C_n are finitely generated, and let $H_n = \ker(d_n)/\operatorname{im}(d_{n+1})$ denote its homology groups. Prove that

$$\sum_n (-1)^n \operatorname{rank}(C_n) = \sum_n (-1)^n \operatorname{rank}(H_n)$$

  (b) The classical definition of the Euler characteristic $\chi(M)$ of a finite cell complex M, is the number of even dimensional cells minus the number of odd dimensional cells. Use the previous part of this problem and the isomorphism between cellular and singular homology to show that

$$\chi(M) = \sum_n (-1)^n \operatorname{rank}(H_n(M)).$$

  Conclude that the Euler characteristic of a cell complex is an invariant of the homotopy type of M (in particular it does not depend on the choice of cell decomposition).

(2) This problem is to calculate the cellular homology of projective spaces.

  (a) The *real n-dimensional projective space*, $\mathbb{R}\mathrm{P}^n$ is the quotient of $\mathbb{R}^{n+1}\backslash 0$ by the equivalence relation $(x_0, x_1, \cdots, x_n) \sim (\lambda x_0, \lambda x_1, \cdots, \lambda x_n)$ for $\lambda \in \mathbb{R}\backslash 0$. Equivalence classes are represented by *homogeneous coordinates*:

$$\mathbb{R}\mathrm{P}^n = \{[x_0 : x_1 : \cdots : x_n] \mid (x_1, x_2, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}\backslash(0, 0, \cdots, 0, 0)\}$$

  where $[x_0 : x_1 : \cdots : x_n] = [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n]$.

    (i) Show that $\mathbb{R}\mathrm{P}^n = X_0 \cup Y_0$ where

$$X_0 = \{[x_0 : x_1 : \cdots : x_n] \mid x_0 = 1\}, \qquad Y_0 = \{[x_0 : x_1 : \cdots : x_n] \mid x_0 = 0\}$$

    and show that X_0 is homeomorphic to $\mathbb{R}^n$ (which itself is homeomorphic to the open n-disk), and that Y_0 is homeomorphic to $\mathbb{R}\mathrm{P}^{n-1}$.

    (ii) Inductively use this to show that $\mathbb{R}\mathrm{P}^n$ has a cell decomposition with one k-cell for $0 \leqslant k \leqslant n$.

    (iii) Calculate the degrees for the gluing maps which attach the k-cell to the $(k-1)$-skeleton.

    (iv) Calculate the cellular homology for $\mathbb{R}\mathrm{P}^n$.

  (b) The *complex n-dimensional projective space*, $\mathbb{C}\mathrm{P}^n$ is defined similarly as the quotient of $\mathbb{C}^{n+1}\backslash 0$ by the equivalence relation $(z_0, z_1, \cdots, z_n) \sim (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$ for $\lambda \in \mathbb{C}\backslash 0$. Using the homogenous coordinates and decompositions analogous to the $\mathbb{R}\mathrm{P}^n$ case, show that $\mathbb{C}\mathrm{P}^n$ has a cell decomposition with one k-cell for each even value of k from 0 to $2n$. Using this cell decomposition, calculate the cellular homology of $\mathbb{C}\mathrm{P}^n$.

(3) Construct a surjective map $S^n \to S^n$ of degree zero for every $n \geqslant 1$.

(4) Compute the homology groups of the following 2-complexes. Clearly explain the cell decomposition you are using and calculate the associated degrees of gluing maps carefully.

(a) $S^1 \times (S^1 \vee S^1)$

(b) The quotient of $S^1 \times S^1$ by identifying the points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying the points on the circle $\{x_0\} \times S^1$ that differ by a $2\pi/n$ rotation.

(5) Let $X = T^2 \times I/\sim$ (here $T^2 = S^1 \times S^1$ is the torus and $I = [0,1]$) where the equivalence relation $\sim$ is defined by $(p, 0) \sim (f(p), 1)$ for every $p \in T^2$ where $f : T^2 \to T^2$ is the homeomorphism defined by identifying $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and letting $f([(x, y)]) = [(x, nx + y)]$ where $n \in Z$. Calculate the homology groups of X.