Even problems

2. Example : $\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ but $-1 \cdot 1 = -1 \neq 0$

4. Example: $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ but $-1 + 0^2 = -1 \neq 0$

6. Given that $(1,2,5)$ $(0,4,1)$ are in column space

assume that our matrix is of form $\begin{pmatrix} 1 & 0 & x_1 \\ 2 & 4 & x_2 \\ 5 & 1 & x_3 \end{pmatrix}$

then by $(1,-1,2)$ in null space, $\begin{pmatrix} 1 & 0 & x_1 \\ 2 & 4 & x_2 \\ 5 & 1 & x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -2 \end{pmatrix}$

So one example is $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 2 & 4 & 1 \\ 5 & 1 & -2 \end{pmatrix}$

8. This is to find all solutions of $A\vec{x} = 0$ and then find basis

Get x_1, x_2 as free variable :

$x_1 = 1 \quad x_2 = 0 \Rightarrow \begin{matrix} 1 + x_3 + 3x_4 = 0 \\ 2 + x_3 + x_4 = 0 \end{matrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$x_1 = 0 \quad x_2 = 1 \Rightarrow \begin{matrix} -2 + x_3 + 3x_4 = 0 \\ -4 + x_3 + x_4 = 0 \end{matrix} \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix}$

So $N(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix} \right.$

10. First do elimination

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 2 & 2 & 3 & 5 & 7 \\ 1 & 1 & 4 & 6 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 5 & 7 & 7 \\ 0 & 0 & 5 & 7 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 5 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

One solution is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $A\vec{x} = 0$ has solution $t_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 2 \\ 7 \\ -5 \end{pmatrix}$

(set x_1, x_3 as free variable, $x_3 = 7$ for convenience

So general solution is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 2 \\ 7 \\ -5 \end{pmatrix}$

12. Elimination is done

One solution of $A\vec{x}=\vec{b}$ is $\begin{pmatrix} 25/3 \\ 0 \\ 17/3 \\ -2 \end{pmatrix}$

$$\begin{cases} x_1+2x_2-x_3+3x_4=0 \\ 3x_3+5x_4=7 \\ -2x_4=4 \end{cases} \quad \begin{cases} x_1=6+\frac{17}{3}-2x_2 \\ x_3=17/3 \\ x_4=-2 \end{cases}$$

Particular solution $x_2=0$

choose free variable

General solution of $A\vec{x}=0$ is $t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

So all solutions are $t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} 25/3 \\ 0 \\ 17/3 \\ -2 \end{pmatrix}$

14. They are linearly independent

suppose $c_1v_1+c_2v_2+c_3v_3=0$ then $\begin{cases} c_1+c_2=0 \\ -c_1+2c_2+2c_3=0 \\ c_2=0 \end{cases}$ only has zero solution

16. They are linearly dependent

$-2v_1+3v_2+(-2)v_3+v_4=0$

18. $C(A)=\text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix},\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ dim=3

$N(A)=\text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right\}$ dim=1

$3+1=4$

So rank nullity thm is satisfied

20. $C(A)=\text{span}\left\{\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix},\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}\right\}$ dim=2

$N(A)=\text{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$ dim=1

$2+1=3$

Rank nullity thm is satisfied

22. $U=\text{span}\left\{\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}\right\}$

24. ST = Sometimes true T = always true F = always False

(a) ST take $v_1 = v_2 = \dots = v_5$

(b) ~~F~~ ST

(c) ST same as (a)

(d) ST same as (a)

(e) ST same as (a)

(f) T

(g) ST Same as (a)

(h) ST Same as (a)

(i) ST U can have v_6 linearly independent from $v_1, \dots, v_5$

(j) ST Same as (i)

(k) ST same as (i)

(l) T

(m) ST same as (i)

(n) T

(o) T

(p) ~~F~~ T ~~take $U = \mathbb{R}^5$ $v_1 \cdots v_5$ to be unit vectors~~

(q) ~~F~~ T ~~same as (p)~~

(r) T

(s) ST same as (a)

(t) ST same as (a)

(u) F Same as (a)

(v) ST same as (a)

(w) ST $U = \mathbb{R}^4$ $v_1 \cdots v_4$ unit vectors $v_5 = 0$.

(x) ST same as (a)

(y) F $U = \mathbb{R}^6$ $v_1 \cdots v_5$ first five unit vectors

(z) T

26. $\dim(C(A)) = 2 \quad \dim(N(A)) = 5$

28. $\dim(C(A)) = 3 \quad \dim(N(A)) = 3$

30. $N(A) = \text{span}\left\{\begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}\right\}$ $\quad \dim = 1$

$C(A^T) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}\right\}$ $\quad \dim = 3$

all 3 inner products give 0

32. $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ is one of the vectors

34. $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is one of the vectors

36. $u_1 \perp u_2 \quad u_1 \perp u_5$

$u_3 \perp u_4 \quad u_3 \perp u_6$

(u_1, u_2) (u_3, u_4) are orthogonal complements

38. $P = \dfrac{ww^T}{w^Tw} = \begin{pmatrix} 9 & 6 & -6 & 6 & 6 \\ 6 & 4 & -4 & 4 & 4 \\ -6 & -4 & 4 & -4 & -4 \\ 6 & 4 & -4 & 4 & 4 \\ 6 & 4 & -4 & 4 & 4 \end{pmatrix} \Big/ (3^2+2^2+(-2)^2+(-2)^2+(-2)^2)$

$P\vec{v} = \dfrac{1}{25}\begin{pmatrix} 3 \\ 2 \\ -2 \\ 2 \\ 2 \end{pmatrix}$

$\swarrow A$

40. $P = \dfrac{ww^T}{w^Tw} = \begin{pmatrix} 9 & -3 & 3 & 6 & 3 & -9 \\ -3 & 1 & -1 & -2 & -1 & 3 \\ 3 & -1 & 1 & 2 & 1 & -3 \\ 6 & -2 & 2 & 4 & 2 & -6 \\ 3 & -1 & 1 & 2 & 1 & -3 \\ -9 & 3 & -3 & -6 & -3 & 9 \end{pmatrix} \Big/ (3^2+(-1)^2+1^2+2^2+1^2+(-3)^2) = \dfrac{1}{25}(A)$

42. $A=\begin{pmatrix}1&1\\1&4\\1&2\end{pmatrix}$ $\vec{b}=\begin{pmatrix}1\\0\\2\end{pmatrix}$

Solve $A^TA\begin{pmatrix}C\\D\end{pmatrix}=A^T\vec{b} \Rightarrow \begin{pmatrix}3&7\\7&21\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$

$\Rightarrow \begin{pmatrix}C\\D\end{pmatrix}=\begin{pmatrix}2\\-\frac{3}{7}\end{pmatrix}$

so the best fit is $y=2-\frac{3}{7}x$

44. Gram-Schmidt:

$u_2=v_2-\mathrm{proj}_{u_1}(v_2)=\begin{pmatrix}2\\0\\-1\\-2\\3\end{pmatrix}-2\begin{pmatrix}1\\0\\-1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\\1\end{pmatrix}$

$\mathrm{Proj}_{\vec{u}}(\vec{v})=\frac{\langle\vec{v},\vec{u}\rangle}{\langle\vec{u},\vec{u}\rangle}\vec{u}=\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$

$e_1=\frac{1}{2}\begin{pmatrix}1\\0\\-1\\-1\\1\end{pmatrix}$

$e_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\\1\end{pmatrix}$

46. $|\vec{v}_1|=|\vec{v}_2|=|\vec{v}_3|=1$

$\vec{v}_1\cdot\vec{v}_2=0$ $\vec{v}_1\cdot\vec{v}_3=0$ $\vec{v}_2\cdot\vec{v}_3=0$

so $\{v_1,v_2,v_3\}$ gives an orthonormal basis

The solution is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\cancel{\begin{pmatrix}\frac{\sqrt{6}}{3}\\0\\\frac{2\sqrt{3}}{3}\end{pmatrix}}\begin{pmatrix}\frac{\cancel{\sqrt{6}}}{3}\\0\\\frac{2\sqrt{3}}{3}\end{pmatrix}$

48. $P=A(A^TA)^{-1}A^T$ $A=\begin{pmatrix}\frac{2}{3}&-\frac{1}{3}&0\\0&0&0\\\frac{2}{3}&0&-\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\0&-\frac{2}{3}&\frac{2}{3}\end{pmatrix}$

Note that $C(A)$ is orthonormal

$A^TA=I$

$P=AA^T=\begin{pmatrix}\frac{5}{9}&0&\frac{4}{9}&0&\frac{2}{9}\\0&0&0&0&0\\\frac{4}{9}&0&\frac{5}{9}&\frac{2}{9}&0\\0&0&\frac{2}{9}&1&-\frac{4}{9}\\\frac{2}{9}&0&0&-\frac{4}{9}&\frac{8}{9}\end{pmatrix}$

50. $\det A = (-(1\times1)) \cdot (-(1\times8)) = 8$

52. $\det A = (-(-(1\times3))) \cdot (-(2\times2)) = -12$

54. $\det A = -5(k-5)$ so when $k=5$, $\det(A) = 0$.

56. $\det A = \det\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k-2 \end{pmatrix}$ so when $k=2$ $\det(A)=0$

(row operation does not change determinant)